MANJUL BHARGAVA'S FIELDS MEDAL AND BEYOND

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The award of the 2014 Fields Medal to Professor Manjul Bhargava of Princeton University is a crowning recognition for his path-breaking contributions to some of the most famous and difficult problems in Number Theory. The Fields Medal, known as the "Nobel Prize of Mathematics", has an age limit of 40, and therefore recognizes outstanding contributions by young mathematicians who are expected to influence the development of the subject in the years ahead. Indeed when the Fields Medals were instituted in 1936, it was stressed that not only were these to recognise young mathematicians for what they had achieved, but also to encourage them to continue to influence the progress of the discipline in the future, and to inspire researchers around the world to collaborate with them or work on the ideas laid out by them to shape the development of mathematics. Manjul Bhargava was awarded the Fields Medal for his pioneering work on several long standing and important problems in number theory, and for the methods he has introduced to achieve this phenomenal progress - methods that will influence research for years to come. Here I will focus on his recent work and its future influence, but to provide a background, I will first briefly describe work stemming from his PhD thesis.

Bhargava's PhD thesis concerns a problem going back to Carl Friedrich Gauss, one of the greatest mathematicians in history. In the nineteenth century, Gauss had discovered a fundamental composition law for binary quadratic forms which are homogeneous polynomial functions of degree two in two variables. No formula or law of the Gauss type was known for cubic or higher degree forms. Bhargava broke the impasse of 200 years by producing a composition law for cubic and higher degree forms. His elegant solution to the cubic case was inspired by studying slicings of the Rubik's cube. In his PhD thesis of 2001 written under the direction of Professor Andrew Wiles of Princeton University, and in his post-doctoral work, Bhargava established the composition law for cubic forms and extended this to the quartic and the quintic cases as well. He also used these composition laws to settle a number of classical problems in number theory, including some cases of the Cohen-Lenstra-Martinet conjectures on class groups, as well as the determination of the asymptotic number of quartic and quintic fields having bounded discriminant. His work appeared in a series of papers in the Annals of Mathematics published by the Princeton University Press.

Around this time he also got interested in the fundamental problem of determining all universal integral quadratic forms, namely quadratic forms that represent all positive integers. This is a problem that stems from the work of Ramanujan who wrote down 54 examples of such universal quadratic forms. Bhargava solved two fundamental problems on universal quadratic forms (the second jointly with Jonathan Hanke), and announced these for the first time in his Ramanujan Commemmoration Lecture at SASTRA University, Kumbakonam, on December 22, 2005 (Ramanujan's birthday) when he was awarded the First SASTRA Ramanujan Prize. This prize has a stricter age limit of 32 because Ramanujan achieved so much in his brief life of 32 years. I should point out that only in 2005 were two SASTRA Ramanujan Prizes awarded - one to Bhargava and another to

Kannan Soundararajan; they were two full prizes, and not shared. For more details about this early work of Bhargava, see my article in The Hindu, Dec. 23, 2005.

Bhargava's work on quadratic and higher degree forms brought a resurgence of activity worldwide on these topics. In view of this, three major conferences were organised in 2009 funded by the National Science Foundation : A conference on quadratic forms and another on higher degree forms at the University of Florida in March and May, and The Arizona Winter School on Quadratic Forms in March. Bhargava acted as a co-organiser for the Florida conferences with me and my colleagues, and was a lead speaker at the Arizona conference. Selected lectures from these conferences have appeared recently as a volume in the book series Developments in Mathematics published by Springer with Bhargava co-editing this volume with me and David Savitt and Pham Tiep of the University of Arizona.

For his seminal contributions on composition laws and on quadratic and higher degree forms, Bhargava received several awards in addition to the SASTRA Ramanujan Prize, such as the Clay Research Award in 2005, and the Frank Nelson Cole Prize of the American Mathematical Society in 2008. As Professor Peter Sarnak of Princeton University said "For a guy so young, I can't remember anybody so decorated at his age". But Bhargava did not rest on his laurels. He continued to pursue fundamental questions with unidiminished zeal. His next assault was on the important problem concerning ranks of elliptic curves and the famous Birch– Swinnerton-Dyer (BSD) Conjecture, which is one of the Millennium Prize Problems.

An *Elliptic Curve* is given by an equation $Y^2 = AX^3 + BX^2 + CX + D$. By change of variables this can be reduced to the form $y^2 = x^3 + ax + b$. If instead we set y^2 equal to a polynomial in x of higher degree, we get a hyper-elliptic curve. Both elliptic and hyper-elliptic curves have been studied in detail over the years. But elliptic curves are especially interesting for two reasons: (i) they constitute the smallest degree polynomials for which we do not know how to determine all the rational solutions, and (ii) they are the only case where the set of rational solutions has a group structure which yields a procedure to produce a third solution given any two.

The BSD Conjecture gives an algorithm to find all rational solutions of an elliptic curve; this algorithm has worked on all elliptic curves on which it has been tested, but it is not been proven that it will work on all elliptic curves, and whether it will give all the rational solutions. It was established by Mordell in 1922 that that the set of rational solutions for an elliptic curve is a finitely generated group. This means there is a fine set of elements such that all the elements of the group can be written in terms of this finite set. The group is also Abelian, which means its elements satisfy the commutative relation $p + q = q + p$. A well-known theorem on finitely generated Abelian groups states that any such group can be written as a sum of copies of the set integers $\mathcal{Z}$ (call the free part of the group) and/or the sum of copies of $\mathcal{Z}_m$, the set of the integers modulo m, for certain m (called the torsion part of the group). The *rank* of an elliptic curve is the dimension of the free part of the group, namely the number of copies of $\mathcal{Z}$ in the product. It was shown by Barry Mazur (Harvard) that the torsion part has no more than 16 elements! So the question really centered on whether the rank can be arbitrarily large. The largest rank found so far is 28; this example is due to Noam Elkies (Harvard).

With regard to the distribution of values of the rank, it was conjectured by Dorian Goldfeld (Columbia), and by Nicholas Katz and Peter Sarnak (Princeton), that asymptotically 50% of the time the rank is 0, and 50% of the time the rank is 1. That means that the asymptotic density of elliptic curves with rank greater than one is zero and that the average rank is 1/2. Numerical evidence in the last few years did not support this conjecture on the distribution of the rank. In fact on the basis of the numerical evidence, it was speculated that this conjecture of Goldfeld-Katz-Sarnak might be false.

Assuming the BSD conjecture and the Generalised Riemann Hypothesis (GRH), it was shown by Brumer in 1992 that the average rank is bounded by 2.3. In 2004, Roger Heath-Brown (Oxford), also assuming BSD and GRH, improved Brummer's bound to 2.0. In the last few years, Manjul Bhargava in collaboration with his PhD student Arul Shankar showed that the average rank is less than 1 (less than .885 to be precise) and that a positive proportion of the elliptic curves have rank 0. These two results, proved unconditionally, that is without any hypothesis, are startling! Contrary to what was believed due to numerical evidence, Bhargava and Shankar's results seemed to indicate that the conjecture that the average rank is 1/2 might be true after all. Bhargava spoke about this tremendous progress in his Plenary Lecture at the conference in New Delhi in December 2012 for Ramanujan's 125-th birth anniversary. Immediately after that conference, he received the 2012 Infosys Prize in Mathematical Sciences.

Very recently, in collaboration with Christopher Skinner (Princeton), Bhargava showed — using important earlier work of Kolyvagin, and Gross-Zagier — that the average rank is indeed positive; this work will soon appear in the Journal of the Ramanujan Mathematical Society. Yet another recent breakthrough of Bhargava is his result with Skinner and Wei Zhang (winner of the 2010 SASTRA Ramanujan Prize) that the BSD Conjecture holds for at least 66% of the elliptic curves. Bhargava has recently taken up the study of hyper-elliptic curves and has shown in collaboration with Benedict Gross (Harvard) and in conjunction with the work of Poonen and Stoll that most hyper-elliptic equations of the form $y^2 = P(x)$, where P is a polynomial of degree at least 6, have no rational solutions. Thus Bhargava's work is making deep inroads in algebraic number theory. The Fields Medal to Bhargava recognises all these pathbreaking contributions to various celebrated problems in number theory, as well the fundamental new methods that he has introduced. He is the second SASTRA Ramanujan Prize Winner to also receive the Fields Medal, the other being Terence Tao in 2006.

In establishing these fundamental results, Bhargava developed several new methods, for instance in the area of mathematics called "the geometry of numbers". A central problem in the geometry of numbers concerns counting lattice points in regions of two dimensional or higher dimensional space. If such a region is bounded, the number of lattice points is approximated by the area or volume of the region. But if the region is unbounded, then the problem is quite complicated; Bhargava has developed new techniques to deal with such unbounded regions. His theorems on counting number fields and studying rational solutions of hyper-elliptic equations have made key use of these new techniques in the geometry of numbers. These techniques are expected to have further applications not just in number theory but in other areas such as topology and knot theory.

Bhargava was made full professor at Princeton soon after his PhD, and at 28 was one of the youngest to achieve that high rank in the history of that great centre of learning. He has a genial personality. That, combined with his reputation, attracted a number of talented students. Some of these students have themselves made a mark and obtained appointments at leading universities. Arul Shankar who graduated from the Chennai Mathematics Institute, went to Princeton to do his PhD under Bhargava's guidance. Shankar is now at Harvard. Melanie Wood, another former PhD student of Bhargava, is now at the University of Wisconsin after having held a post-doctoral position at Stanford. Wei Ho who obtained her PhD under Bhargava's supervision, is now at the University of Michigan after holding post-doctoral positions at Harvard and Columbia.

In addition to guiding PhD students and mentoring post-docs in Princeton, Bhargava visits India regularly, especially the Tata Institute, IIT Bombay, and the University of Hyderabad, where he holds Adjunct Professorships. He has a deep regard for Indian culture and is proud of his Indian heritage. He enjoys working with mathematicians in India. For example, he has an active collaboration with Eknath Ghate of the Tata Institute who recently received the Bhatnagar Award. Bhargava's profound influence on mathematics is not only due to his own fundamental contributions, but also due to the students and post-docs he has groomed, and the impact he has had on the work of the present generation of mathematicians. This influence will continue strongly in the years ahead and consequently will lead to more breakthroughs in the future. Perhaps we will see the resolution of the BSD Conjecture!