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# Duality between prime factors and the prime number theorem for arithmetic progressions—II

**Krishnaswami Alladi[1] · Jason Johnson[1]**

*Dedicated to George Andrews and Bruce Berndt for their 85th birthdays*

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**Abstract**

In the first paper under this title (1977), the first author utilized a duality identity between the largest and smallest prime factors involving the Moebius function, to establish the following result as a consequence of the prime number theorem for arithmetic progressions: If $k$ and $\ell$ are positive integers, with $1 \le \ell \le k$ and $(\ell, k) = 1$, then

$$\sum_{n\ge 2,\ p_1(n)\equiv \ell (mod\, k)} \frac{\mu(n)}{n} = \frac{-1}{\phi(k)},$$

where $\mu(n)$ is the Moebius function, $p_1(n)$ is the smallest prime factor of $n$, and $\phi(k)$ is the Euler function. Here we utilize the next level Duality identity between the second largest prime factor and the smallest prime factor, involving the Moebius function and $\omega(n)$, the number of distinct prime factors of $n$, to establish the following result as a consequence of the prime number theorem for arithmetic progressions: For all $\ell$ and $k$ as above,

$$\sum_{n\ge 2,\ p_1(n)\equiv \ell (mod\, k)} \frac{\mu(n)\omega(n)}{n} = 0.$$

A quantitative version of this result is proved.

**Keywords** Duality between prime factors · Moebius function · Number of prime factors · Smallest prime factor · Largest prime factor · Second largest prime factor · Prime number theorem for arithmetic progressions · Extension to number fields

✉ Krishnaswami Alladi
alladik@ufl.edu

Jason Johnson
iridiumalchemist@gmail.com

1 Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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## 1 Introduction and statement of the main theorem

Two famous results of Edmund Landau are that

$$M(x) := \sum_{1\le n\le x} \mu(n) = o(x), \quad \text{as} \quad x \to \infty \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \tag{1.2}$$

are (elementarily) equivalent to the prime number theorem (PNT), where $\mu(n)$ is the Moebius function. Similarly, there are results equivalent to the prime number theorem for arithmetic progressions (PNTAP) in which $\mu(n)$ is replaced by $\mu(n)\chi(n)$, where $\chi(n)$ is a Dirichlet character modulo $k$, when the arithmetic progression under consideration has common difference $k$.

In [2], the first author noticed the following interesting Duality identities involving the Moebius function that connect the smallest and largest prime factors of integers:

$$\sum_{2\le d|n} \mu(d) f(p_1(d)) = -f(P_1(n)), \tag{1.3}$$

and

$$\sum_{2\le d|n} \mu(d) f(P_1(d)) = -f(p_1(n)), \tag{1.4}$$

where for $d > 1$, $p_1(d)$ and $P_1(d)$ denote the smallest and largest prime factors of $d$ respectively, and $f$ is ANY function on the primes. Using (1.3) and properties of the Moebius function, it was shown in [2] that if $f$ is a bounded function on the primes such that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{2\le n\le x} f(P_1(n)) = c, \tag{1.5}$$

then

$$\sum_{n=2}^{\infty} \frac{\mu(n) f(p_1(n))}{n} = -c, \tag{1.6}$$

and vice-versa. This is a surprising generalization of Landau's result (1.2). To realize this is a generalization, rewrite (1.2) as

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} = -1. \tag{1.7}$$

Then (1.7) follows from (1.5) and (1.6) by taking $f(p) = 1$ for all primes $p$.

Next it was shown in [2] that the PNTAP implies that the sequence $P_1(n)$ of largest prime factors is uniformly distributed in the reduced residue classes modulo a positive integer $k$. So if $f$ is chosen to be the characteristic function of primes in an arithmetic progression $\ell (mod\, k)$, then for such $f$, (1.5) holds with $c = 1/\phi(k)$, and therefore

$$\sum_{n\geq 2,\ p_1(n)\equiv \ell (mod\, k)} \frac{\mu(n)}{n} = \frac{-1}{\phi(k)}, \tag{1.8}$$

for ALL positive integers $k$ and any $\ell$ satisfying $(\ell, k) = 1$. This is even more surprising because it gives a way of slicing the convergent series in (1.7) into $\phi(k)$ subseries all converging to the same value! As far as we know, this is the first example of slicing convergent series into equal valued subseries. In the last few years, (1.8) has been generalized in the setting of algebraic number theory (see [6, 9, 12, 17, 22, 23]).

In [2], the following more general duality identities were noted: For a positive integer $r$, let $P_r(n)$ and $p_r(n)$ denote the $r$-th largest and $r$-th smallest prime factors $n$ respectively (defined by strict inequalities), if $n$ has at least $r$ distinct prime factors. Also let $\omega(n)$ denote the number of distinct prime factors of $n$. Then

$$\sideset{}{^*}\sum_{1<d|n} \mu(d) f(P_r(d)) = (-1)^r \binom{\omega(n)-1}{r-1} f(p_1(n)), \tag{1.9}$$

and

$$\sideset{}{^*}\sum_{1<d|n} \mu(d) f(p_r(d)) = (-1)^r \binom{\omega(n)-1}{r-1} f(P_1(n)), \tag{1.10}$$

where the * over the summation means that if $n$ has fewer than $r$ distinct prime factors, then the sum is zero. From (1.9) and (1.10), it follows by Moebius inversion that

$$\sum_{1<d|n} \mu(d) \binom{\omega(d)-1}{r-1} f(P_1(d)) = (-1)^r f(p_r(n)), \tag{1.11}$$

and

$$\sum_{1<d|n} \mu(d) \binom{\omega(d)-1}{r-1} f(p_1(d)) = (-1)^r f(P_r(n)). \tag{1.12}$$

In these identities, we adopt the convention that for an integer $m$, if $\omega(m) < r$, then $f(P_r(m)) = f(p_r(m)) = 0$.

In this paper we will discuss consequences of (1.12) in the case $r = 2$, that is

$$\sum_{1<d|n} \mu(d)(\omega(d)-1) f(p_1(d)) = f(P_2(n)) \tag{1.13}$$

and its implications.

Analogous to (1.2), we establish (see Theorem 4 (i) of Sect. 2) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)\omega(n)}{n} = \sum_{n=2}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0, \tag{1.14}$$

where the first equality in (1.14) holds because $\omega(1) = 0$.

Next, just as we established in [2] the uniform distribution of $P_1(n)$ in the reduced residue classes $\bmod\ k$, we prove in Sect. 4 using the strong form of the PNTAP, the following quantitative form (see Theorem 7 in Sect. 4) of the uniform distribution of $P_2(n)$ in the reduced residue classes $(\bmod\, k)$, for every integer $k$:

*For each fixed* $k \geq 2$, *and any* $1 \leq \ell < k$ *with* $(\ell, k) = 1$, *we have*

$$\sum_{n \leq x,\ P_2(n) \equiv \ell (mod\ k)} 1 = \frac{x}{\phi(k)} + O\Big(\frac{x(\log\log x)^2}{\log x}\Big). \tag{1.15}$$

This uniform distribution property of $P_2(n)$ is crucial in our approach to establish the main result (Theorem 10 in Sect. 5) which is:

*For integers* $\ell, k$ *satisfying* $1 \leq \ell \leq k$ *and* $(\ell, k) = 1$, *we have*

$$m_\omega(x; \ell, k) := \sum_{n \leq x,\ p_1(n) \equiv \ell (mod\, k)} \frac{\mu(n)\omega(n)}{n} << \frac{(\log\log x)^{5/2}}{\sqrt{\log x}}. \tag{1.16}$$

*Consequently, with* $\ell, k$ *as above, we have*

$$\sum_{n \geq 2,\ p_1(n) \equiv \ell (mod\, k)} \frac{\mu(n)\omega(n)}{n} = 0. \tag{1.17}$$

So we have sliced the series in (1.14) into $\phi(k)$ subseries all convergent to 0.

This main result is established using the uniform distribution of $P_2(n)$ and the duality identity (1.13), with $f(p)$ chosen to be the characteristic function of primes $p$ in the arithmetic progression $\ell(mod\, k)$. To establish (1.15)–(1.17), several auxiliary results are proved.

**Notations and Conventions:** In what follows, $c, c_1, c_2, \ldots$ are absolute positive constants whose values will not concern us. The $<<$ and $O$ notations are equivalent and will be used interchangeably as is convenient. We also adopt the convention that

$$f(x) \ll g(x) \quad \text{means} \quad |f(x)| < K|g(x)|,$$

with $x$ ranging in some domain, and $K$ a positive constant. All asymptotic estimares and bounds involving $x$ will be valid for $x \geq x_0$. Implicit constants are absolute unless otherwise indicated. Although our results can be established with uniformity by allowing the modulus $k$ to grow slowly as a function of $x$, we only consider here an arbitrary but fixed modulus $k$. The alphabet $n$ whether used as the argument of a function, or in a summation, will always be a positive integer. Also, any time we have a sum over $p$, or have $p$ as an argument of a function, it is to be understood that $p$ is prime.

By $E(x, k, \ell)$ we mean the difference

$$E(x, k, \ell) = \pi(x, k, \ell) - \frac{\ell i(x)}{\phi(k)},$$

where

$$\pi(x, k, \ell) = \sum_{p \leq x,\, p \equiv \ell (mod\, k)} 1, \quad \text{and} \quad \ell i(x) = \int_2^x \frac{dt}{\log t}.$$

When $k$ and $\ell$ are specific, we simply use $E(x)$ in place of $E(x, k, \ell)$. Finally, by $R(x)$ we mean any decreasing function of $x$ that tends to zero as $x \to \infty$ and bounds from above the relative error in the PNTAP; that is

$$\left| \pi(x, k, \ell) - \frac{\ell i(x)}{\phi(k)} \right| < \frac{\ell i(x)}{\phi(k)} R(x).$$

In what follows, we will choose

$$R(x) = e^{-c\sqrt{\log x}},$$

where $c$ can be any positive constant. Also $T = T(x)$ will be a function which will be chosen optimally to get suitable bounds in various estimates, but $T$ will not necessarily be the same in different contexts. We shall use the standard notation $[x]$ for the integral part of a real number $x$, and $\{x\}$, where indicated, will denote the fractional part of $x$, namely $x - [x]$. Finally, complex numbers will be denoted by $s = \sigma + it$ when dealing with Dirichlet series. Further notation will be introduced in the sequel as needed.

## 2 The Moebius function and the number of prime factors

The Dirichet series associated with the Moebius function is

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad \text{for} \quad \sigma > 1, \tag{2.1}$$

where $\zeta(s)$ is the Riemann zeta function. By using bounds for $1/\zeta(s)$, the standard analytic approach to obtain the strong form of the PNT can be employed to obtain the following bound for $M(x)$—see for instance, Tenenbaum [19, p. 217]:

$$M(x) = \sum_{n \leq x} \mu(n) << x e^{-c_1\sqrt{\log x}}. \tag{2.2}$$

The same method also yields

$$m(x) := \sum_{n\le x} \frac{\mu(n)}{n} << e^{-c_1\sqrt{log\, x}}, \tag{2.3}$$

which is a quantitative form of (1.2). In (2.2) and (2.3), we have used the same constant $c_1$, because if we had two different positive constants, we could choose the minimum of these two as $c_1$ for (2.2) and (2.3) to hold. In addition to (2.2) and (2.3), we have for each positive integer $j$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)\log^j n}{n} = \lambda_j \tag{2.4}$$

is convergent and its quantitative form

$$\sum_{n\le x} \frac{\mu(n)\log^j n}{n} = \lambda_j + O(e^{-c_1\sqrt{log\, x}}). \tag{2.5}$$

The constant $c_1 = c_1(j)$ in the exponential in (2.5) depends on $j$, but in what follows, we will use (2.5) only for $j \le \nu$, where $\nu$ will be an arbitrary but fixed positive integer. So we will use

$$c_1 := \min\{c_1(0), c_1(1), \ldots, c_1(\nu)\}$$

in (2.2), (2.3). and (2.5). Note that (2.5) can be deduced from the method that yields (2.3) or by partial summation using (2.3). It is known that

$$\sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} = \lambda_1 = -1. \tag{2.6}$$

This is the only value besides $\lambda_0 = 0$ that we need. The actual values of $\lambda_j$ for $j \ge 2$ which can be written in terms of the successive derivative values of $\zeta(s)^{-1}$ at $s = 1$, will not concern us.

Even though our focus is on square-free $n$, in order to employ the Moebius inversion formula, it is convenient to also consider the function $\Omega(n)$, which is the (total) number of prime factors of $n$ counted with multiplicity, because the $\Omega(n)$ is a totally additive.

If $\chi_P$ denotes the characteristic function of the prime powers, then

$$\Omega(n) = \sum_{d|n} \chi_P(d).$$

Thus by Moebius inversion, we have

$$\chi_P(n) = \sum_{d|n} \mu(d)\Omega\Big(\frac{n}{d}\Big) = \Omega(n)\sum_{d|n}\mu(d) - \sum_{d|n}\mu(d)\Omega(d) = -\sum_{d|n}\mu(d)\Omega(d),$$

because $\Omega(n)\sum_{d|n}\mu(d)$ is identically zero. Since $\mu(d)=0$ if $d$ is not square-free, we may rewrite this as

$$\sum_{d|n}\mu(d)\omega(d)=-\chi_P(n), \tag{2.7}$$

and Moebius inversion applied to (2.7) yields

$$\mu(n)\omega(n)=-\sum_{d|n}\chi_P(d)\mu\Big(\frac{n}{d}\Big). \tag{2.8}$$

The first result is:

**Theorem 1** (Tenenbaum) *Let*

$$M_\omega(x):=\sum_{n\le x}\mu(n)\omega(n).$$

*Let $\nu$ be an arbitrary but fixed positive integer. Then there exist constants $\beta_j$, such that*

$$M_\omega(x)=\frac{x}{\log^2 x}+\frac{\beta_3 x}{\log^3 x}+\frac{\beta_4 x}{\log^4 x}+\cdots+\frac{\beta_\nu x}{\log^\nu x}+O_\nu\Big(\frac{x(\log\log x)^{2\nu+2}}{\log^{\nu+1} x}\Big).$$

□

***Remark*** Previously, by an elementary method, we had established the bound

$$M_\omega(x) << \frac{x}{\log x} \tag{2.9}$$

using the strong form of the PNT. After the first author communicated this bound to Tenenbaum, along with Theorem 4 (i) below as a consequence, he responded in a letter [20], that the precise expansion for $M_\omega(x)$ given here as Theorem 1, follows from Theorem 11.5.2 in his book [19] by the analytic Selberg–Delange method, as does Theorem 4 (ii). Tenenbaum had a different, but equivalent form for the "O" term in Theorem 1. The referee of this paper suggested that we ought to try the elementary hyperbola method that we have employed in subsequent sections of this paper, to convert the upper bound in (2.9) for $M_\omega(x)$ that we had obtained, into an asymptotic estimate. It turned out that using the hyperbola method, we were able to get a purely elementary proof of Theorem 1 and of Theorem 4 (ii) using the strong form of the PNT, and that is what we provide below because this method is what is used to prove the main Theorems 7 and 10.

***Proof of Theorem 1 (Alladi–Johnson)*** Use (2.8) to write

$$M_\omega(x) = \sum_{n\le x} \mu(n)\omega(n) = -\sum_{n\le x}\sum_{d|n} \mu(d)\chi_P\left(\frac{n}{d}\right). \tag{2.10}$$

Using the hyperbola method, we break up the double sum in (2.10) into

$$M_\omega(x) = -\sum_{m\le T} \mu(m) \sum_{h\le \frac{x}{m}} \chi_P(h) - \sum_{h\le \frac{x}{T}} \chi_P(h) \sum_{T<m\le \frac{x}{h}} \mu(m) = \Sigma_1 + \Sigma_2, \tag{2.11}$$

where $T$ will be chosen optimally below so that

$$\log T = o(\log x). \tag{2.12}$$

We can easily bound $\Sigma_2$ using (2.2) and the monotone increasing property of

$$R_1(x) = xe^{-c_1\sqrt{\log x}}, \quad \text{for} \quad x \ge x_0.$$

That is, (2.2) yields

$$\Sigma_2 << \sum_{h\le \frac{x}{T}} \frac{x\chi_P(h)}{he^{c_1\sqrt{\log(x/h)}}} << \frac{x\log\log x}{e^{c_1\sqrt{\log T}}}. \tag{2.13}$$

The estimation of $\Sigma_1$ is more involved. We use the strong form of the PNT to get

$$\sum_{h\le x} \chi_P(h) = \pi(x) + O(\sqrt{x}) = \ell i(x) + O(xe^{-c\sqrt{\log x}})$$

$$= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots \frac{(\nu-1)!x}{\log^\nu x} + O\left(\nu!\frac{x}{\log^{\nu+1} x}\right), \tag{2.14}$$

by repeated integration-by-parts of $\ell i(x)$. From (2.14) and (2.11) we get

$$\Sigma_1 = -\sum_{m\le T} \mu(m)\left\{\frac{x}{m\log(x/m)} + \frac{x}{m\log^2(x/m)} + \frac{2!x}{m\log^3(x/m)} + \cdots\right.$$

$$\left. + \frac{(\nu-1)!x}{m\log^\nu(x/m)} + O\left(\frac{\nu!x}{m\log^{\nu+1}(x/m)}\right)\right\}$$

$$= -\sum_{m\le T}\left\{\sum_{j=1}^{\nu} \frac{x\mu(m)}{m\log^j(x/m)}\right\} + O\left(\frac{\nu!x\log T}{\log^{\nu+1}(x/T)}\right). \tag{2.15}$$

Denote by

$$\Sigma_{1,j} = -x \sum_{m \le T} \frac{\mu(m)}{m \log^j(x/m)}, \quad \text{for} \quad j = 1, 2, \ldots, \nu. \tag{2.16}$$

We first evaluate $\Sigma_{1,1}$. Note that

$$\Sigma_{1,1} = -x \sum_{m \le T} \frac{\mu(m)}{m \log(x/m)} = -\frac{x}{\log x} \sum_{m \le T} \frac{\mu(m)}{m\left(1 - \frac{\log m}{\log x}\right)}$$

$$= -\frac{x}{\log x} \sum_{m \le T} \frac{\mu(m)}{m} \left\{1 + \frac{\log m}{\log x} + \frac{\log^2 m}{\log^2 x} + \cdots + \frac{\log^{\nu-1} m}{\log^{\nu-1} x} + O_\nu\left(\frac{\log^\nu m}{\log^\nu x}\right)\right\}$$

$$= -\frac{x}{\log x} \sum_{m \le T} \frac{\mu(m)}{m} - \frac{x}{\log^2 x} \sum_{m \le T} \frac{\mu(m) \log m}{m} - \frac{x}{\log^3 x} \sum_{m \le T} \frac{\mu(m) \log^2 m}{m} - \cdots$$

$$- \frac{x}{\log^\nu x} \sum_{m \le T} \frac{\mu(m) \log^{\nu-1} m}{m} + O_\nu\left(\frac{x \log^{\nu+1} T}{\log^{\nu+1} x}\right)$$

$$= O\left(\frac{x}{\log x} e^{-c_1\sqrt{\log T}}\right) + \frac{x}{\log^2 x} - \frac{\lambda_2 x}{\log^3 x} - \cdots - \frac{\lambda_{\nu-1} x}{\log^\nu x} + O_\nu\left(\frac{x \log^{\nu+1} T}{\log^{\nu+1} x}\right), \tag{2.17}$$

in view of (2.3)–(2.6).

Note that the leading term in $\Sigma_{1,1}$ is $x/\log^2 x$. We now obtain a series representation for $\Sigma_{1,j}$ for $j \ge 2$ similar to (2.17) in decreasing powers of $\log x$ up to $\log^{-\nu} x$:

$$\Sigma_{i,j} = -\frac{x}{\log^j x} \sum_{m \le T} \frac{\mu(m)(j-1)!}{m\left(1 - \frac{\log m}{\log x}\right)^j}$$

$$= -\frac{x}{\log^j x} \sum_{m \le T} \left\{\left(\sum_{i=0}^{\nu-j} \frac{(j+i-1)!\mu(m) \log^i m}{i! m \log^i x}\right) + O_\nu\left(\frac{\log^{\nu-j+1} m}{\log^{\nu-j+1} x}\right)\right\}$$

$$= -\sum_{i=0}^{\nu-j} \left\{\frac{(j+i-1)!\lambda_i}{i!} \frac{x}{\log^{j+i} x}\right\} + O\left(\frac{x}{\log^j x} e^{-c_1\sqrt{\log T}}\right) + O_\nu\left(\frac{x \log^{\nu-j+2} T}{\log^{\nu+1} x}\right). \tag{2.18}$$

Since $\lambda_0 = 0$ and $\lambda_1 = -1$, the leading term in the expansion of $\Sigma_{1,j}$ is

$$\frac{j!x}{\log^{j+1} x}.$$

So we see from (2.17) and (2.18) that for each $j \geq 1$ the term $x/\log^{j+1} x$ occurs only in $\Sigma_{1,1}$, $\Sigma_{1,2}$,..., up to $\Sigma_{1,j}$. So we may sum the expressions in (2.17) and in (2.18) up to $j = \nu$ to get

$$\Sigma_1 = \frac{x}{\log^2 x} + \frac{\beta_3 x}{\log^3 x} + \frac{\beta_4 x}{\log^4 x} + \cdots + \frac{\beta_\nu x}{\log^\nu x} + O_\nu\left(\frac{x \log^{\nu+1} T}{\log^{\nu+1} x}\right) + O\left(\frac{x}{\log x} e^{-c_1\sqrt{\log T}}\right), \tag{2.19}$$

where the constants $\beta_j$ are given in terms of the $\lambda_i$ for $i \leq j$. At this stage we choose

$$T = e^{U^2(\log\log x)^2} \quad \text{with} \quad c_1 U = \nu + 1.$$

With this choice of $T$, Theorem 1 follows from (2.19), (2.13) and (2.11). □

For the purpose of proving Theorem 2, also for subsequent use in this paper, and for use in future related works [1, 4], we establish Theorem A below, which is a variant of Axer's theorem (see Hardy [10, p. 378]) for functions which need not be bounded:

**Theorem A** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of reals such that*

$$A(x) := \sum_{n \leq x} a_n \ << \ x\eta(x), \tag{2.20}$$

*where*

$$\eta(x) \to 0 \quad as \quad x \to \infty, \tag{2.21a}$$

*and*

$$x\eta(x) \quad \text{is an increasing function} \quad \to \infty \quad as \quad x \to \infty. \tag{2.21b}$$

*Suppose also that*

$$\sum_{n \leq x} |a_n| << x\beta(x), \tag{2.22}$$

*where*

$$\beta(x) \quad \text{is an increasing function}, \quad \text{and} \quad \beta(x)\eta(x) \to 0 \quad as \quad x \to \infty. \tag{2.23}$$

*Then*

$$\sum_{n \leq x} a_n \left\{\frac{x}{n}\right\} << x\sqrt{\eta(x)\beta(x)} = o(x), \tag{2.24}$$

*where $\{x\}$ is the fractional part of $x$.* □

***Remark*** Axer [5] stated his theorem for bounded functions (sequences) $a_n$, but the method can be applied with $|a_n|$ having at most a slowly growing unbounded average as given by (2.22) and (2.23). In the version of Axer's theorem in Hardy [10, p. 378], the function $\{x\}$ is replaced by a more general function $\chi(x)$ of bounded variation on finite intervals, but $|a_n|$ is assumed to have a bounded average. Theorem A can be generalized by replacing $\{x\}$ with such a function $\chi(x)$, but the version of Theorem A given above suffices for our purpose here and in our subsequent works [1, 4].

***Proof*** Write

$$\sum_{n\leq x} a_n\left\{\frac{x}{n}\right\} = \sum_{n\leq T} + \sum_{T<n\leq x} = \Sigma_3 + \Sigma_4, \tag{2.25}$$

where $T = T(x) \leq x$ will be chosen optimally below. Clearly,

$$|\Sigma_3| \leq \sum_{n\leq T} |a_n| << T\beta(x). \tag{2.26}$$

Next,

$$\Sigma_4 = \sum_{T<n\leq x} (A(n) - A(n-1))\left\{\frac{x}{n}\right\}$$

$$= A([x])\left\{\frac{x}{[x]}\right\} - A([T]-1)\left\{\frac{x}{[T]}\right\} + \sum_{T<n\leq x} A(n)\left(\left\{\frac{x}{n}\right\} - \left\{\frac{x}{n+1}\right\}\right). \tag{2.27}$$

Thus by (2.20), (2.21b), and (2.27), we have

$$|\Sigma_4| = \left|\sum_{T<n\leq x} a_n\left\{\frac{x}{n}\right\}\right| << x\eta(x) + x\eta(x)V_{\{\}}\left[1, \frac{x}{T}\right] << x\eta(x)\frac{x}{T}, \tag{2.28}$$

where $V_{\{\}}[1, \frac{x}{T}]$ is the total variation of the function $\{y\}$ in the interval $[1, \frac{x}{T}]$ and this is $<< \frac{x}{T}$. So from (2.25)–(2.28) we get

$$\sum_{n\leq x} a_n\left\{\frac{x}{n}\right\} = \Sigma_3 + \Sigma_4 << T\beta(x) + \frac{x^2\eta(x)}{T}. \tag{2.29}$$

By choosing

$$T = x\sqrt{\frac{\eta(x)}{\beta(x)}}$$

in (2.29), we get (2.24) and that proves Theorem A. □

**Theorem 2** *With $\{w\}$ denoting the fractional part of $w$, we have*

$$\sum_{n\le x}\mu(n)\omega(n)\{\frac{x}{n}\}\ll\frac{x\sqrt{\log\log x}}{\log x}.$$

□

***Proof*** To prove Theorem 2, choose $a_n=\mu(n)\omega(n)$ in Theorem A. By Theorem 1, we know that in this case we can take

$$\eta(x)=\frac{1}{\log^2 x}$$

in (2.20). Thus (2.21a) and (2.21b) are satisfied. In view of the simple estimate

$$\sum_{1\le n\le x}\omega(n)=\sum_{1\le n\le x}\sum_{p|n}1=\sum_{p\le x}\sum_{n\le x,n\equiv 0(mod\ p)}1\le\sum_{p\le x}\frac{x}{p}\ll x\log\log x,$$

we can take

$$\beta(x)=\log\log x$$

in (2.22), since $|a_n|\le\omega(n)$. Thus (2.23) is satisfied. Theorem 2 then follows from (2.24) of Theorem A. □

Since Theorem 2 deals with the fractional part function as the weight, we establish next the corresponding result with the weight as the integral part function:

**Theorem 3** *Let $[w]$ denote the integral part of $w$. Then for each positive integer $\nu$ we have*

$$\sum_{n\le x}\mu(n)\omega(n)[\frac{x}{n}]=-\frac{x}{\log x}-\frac{x}{\log^2 x}-\frac{2!x}{\log^3 x}-\cdots-\frac{(\nu-1)!x}{\log^\nu x}+O\Big(\nu!\frac{x}{\log^{\nu+1}x}\Big).$$

□

***Proof*** Note that (2.8) yields

$$\sum_{n\le x}\mu(n)\omega(n)[\frac{x}{n}]=\sum_{n\le x}\sum_{d|n}\mu(d)\omega(d)=-\sum_{n\le x}\chi_P(n).$$

and from this Theorem 3 follows using (2.14). □

We now prove the main result of this section which we state in two parts. The reason for this split will be clear at the end of the proof and from the Remark below:

**Theorem 4** *(i) We have*

$$m_\omega(x) := \sum_{n\le x}\frac{\mu(n)\omega(n)}{n} = O\Big(\frac{\sqrt{\log\log x}}{\log x}\Big).$$

*Consequently*

$$\sum_{n=1}^{\infty}\frac{\mu(n)\omega(n)}{n} = \sum_{n=2}^{\infty}\frac{\mu(n)\omega(n)}{n} = 0.$$

*(ii)* (**Tenenbaum**) *More precisely, there exist constants* $\gamma_1, \gamma_2, \gamma_3, \ldots$, *such that for each positive integer* $\nu$ *we have*

$$m_\omega(x) = -\frac{1}{\log x} + \frac{\gamma_2}{\log^2 x} + \frac{\gamma_3}{\log^3 x} + \cdots + \frac{\gamma_\nu}{\log^\nu x} + O_\nu\Big(\frac{(\log\log x)^{2\nu+4}}{\log^{\nu+1} x}\Big).$$

□

***Proof of (i):*** From Theorems 2 and 3, we get

$$\sum_{n\le x}\mu(n)\omega(n)\frac{x}{n} = \sum_{n\le x}\mu(n)\omega(n)\Big[\frac{x}{n}\Big]+\sum_{n\le x}\mu(n)\omega(n)\Big\{\frac{x}{n}\Big\} << \frac{x\sqrt{\log\log x}}{\log x}+\frac{x}{\log x}. \tag{2.30}$$

By cancelling $x$ on both extremes of (2.30), we get the bound for $m_\omega(x)$ in Theorem 4 (i). By letting $x \to \infty$ in this bound, we get the second assertion of Theorem 4 (i).

□

***Proof of (ii) (Alladi–Johnson):*** We establish Theorem 4 (ii) using Theorem 1 and partial summation, but we need the second assertion of Theorem 4 (i) to complete the proof.

We start with the representation

$$m_\omega(x) = \sum_{n\le x}\frac{\mu(n)\omega(n)}{n} = \int_1^x \frac{dM_\omega(t)}{t}, \tag{2.31}$$

Note that $M_\omega(t) = m_\omega(t) = 0$ for $t < 2$. Integration-by-parts of the Stietjes integral in (2.31) gives

$$m_\omega(x) = \frac{M_\omega(t)}{t}\Big|_1^x + \int_1^x \frac{M_\omega(t)}{t^2}dt = \frac{M_\omega(x)}{x} + \int_1^x \frac{M_\omega(t)}{t^2}dt. \tag{2.32}$$

We know from Theorem 4 (i) that $m_\omega(\infty) = 0$, and from Theorem 1 that $M_\omega(x) = o(x)$. So by letting $x \to \infty$ in (2.32), we deduce that

$$\int_1^\infty \frac{M_\omega(t)}{t^2}dt = 0. \tag{2.33}$$

In view of (2.33), we may rewrite (2.32) as

$$m_\omega(x) = \frac{M_\omega(x)}{x} - \int_x^\infty \frac{M_\omega(t)}{t^2} dt. \tag{2.34}$$

Next we insert in (2.34) the expansion for $M_\omega$ given in Theorem 1, but with $\nu$ replaced by $\nu + 1$ in the expansion for $M_\omega(t)$ in the integral. This gives

$$m_\omega(x) = \Big\{\frac{1}{\log^2 x} + \frac{\beta_3}{\log^3 x} + \frac{\beta_4}{\log^4 x} + \cdots + \frac{\beta_\nu}{\log^\nu x} + O_k\Big(\frac{\log\log^{2\nu+2} x}{\log^{\nu+1} x}\Big)$$

$$- \sum_{j=2}^{\nu+1} \beta_j \int_x^\infty \frac{1}{t \log^j t} dt + O\Big(\int_x^\infty \frac{\log\log^{2\nu+4} t}{t \log^{\nu+2} t} dt\Big), \tag{2.35}$$

where we could set $\beta_2 = 1$. Note that

$$\int_x^\infty \frac{dt}{t \log^j t} = \frac{1}{(j-1)\log^{j-1} x}, \quad \text{for each} \quad j \geq 2. \tag{2.36}$$

Also,

$$\int_x^\infty \frac{\log\log^{2\nu+4} t}{t \log^{\nu+2} t} dt << \frac{\log\log^{2\nu+4} x}{\log^{\nu+1} x}. \tag{2.37}$$

So from (2.35), (2.36) and (2.37), we get Theorem 4 (ii) with

$$\gamma_j = \beta_j - \frac{\beta_{j+1}}{j}, \quad \text{for} \quad j \geq 2. \tag{2.38}$$

This completes the proof of Theorem 4 (ii). □

NOTE: Since the term $x/\log x$ does not exist in the expansion of $M_\omega(x)$ in Theorem 1, we could formally set $\beta_1 = 0$. In that case, (2.37) will hold for $j \geq 1$.

***Remark*** From Theorem 1 it follows that

$$M_\omega(x) \sim \frac{x}{\log^2 x}, \quad \text{as} \quad x \to \infty. \tag{2.39}$$

Tenenbaum [20] notes that with partial summation one gets from (2.39) that

$$\sum_{n=0}^{\infty} \frac{\mu(n)\omega(n)}{n} = c_2 \quad \text{(converges)}. \tag{2.40}$$

He then shows that $c_2 = 0$ by considering the limit of a certain analytic function. Tenenbaum then gives the series expansion for $m_\omega(x)$ that we have stated in Theorem 4 (ii). In contrast, we deduced that $c_2 = 0$ because we proved Theorem 4 (i) using Theorem A.

We point out that such precise results for $M_\omega(x)$ and $m_\omega(x)$ are not needed in later sections of this paper because bounds for $M_\omega(x)$ and $m_\omega(x)$ are not used to prove our main result (Theorem 10). Section 2 is included here to provide the context and motivation for Theorem 10, and also because it is this elementary method of proof of Theorems 1 and 4 that is used in the later sections of this paper, and in subsequent works [1, 4]. Also, Sengupta [16] has employed this elementary method in extending the results of this paper to the setting of Galois extensions of the rationals.

## 3 The sizes of the largest and second largest prime factors

The fundamental counting function associated with the largest prime factor $P_1(n)$ is

$$\Psi(x, y) = \sum_{n \le x. P_1(n) \le y} 1. \tag{3.1}$$

Here and in what follows, we shall denote by $u$, the quantity $\frac{\log x}{\log y}$. In an important paper [8], de Bruijn showed that with some constant $c > 0$,

$$\Psi(x, y) << x e^{-cu}, \quad \text{uniformly for} \quad 2 \le y \le x. \tag{3.2a}$$

Tenenbaum [19, Theorem III.5.1] has shown that (3.2a) holds with $c = 1/2$. de Bruijn also proved that

$$\Psi(x, y) << x \log^2 y e^{-u \log u - u \log\log u + O(u)}, \quad \text{for} \quad y > \log^2 x, \tag{3.2b}$$

and indeed the uniform asymptotic estimate

$$\Psi(x, y) \sim x\rho(u), \quad \text{for} \quad e^{(\log x)^{3/5}} \le y \le x, \tag{3.2c}$$

where $\rho$ satisfies the integro-difference equation

$$\rho(u) = 1 - \int_1^u \frac{\rho(v-1)dv}{v} \tag{3.3a}$$

and (de Bruijn [7])

$$\rho(u) = e^{-u \log u - u \log\log u + O(u)}. \tag{3.3b}$$

Thus $\Psi(x, y)$ is quite small in comparison with $x$ when $u$ is large.

***Remark*** Note that (3.2b) is of no use when $u > 1$ is fixed, because trivially $\Psi(x, y) \le x$; so (3.2b) is used only when $u \to \infty$ with $x$. Much better extimates for $\Psi(x, y)$ are known including those that significantly extend the range of the asymptotic formula (3.2c); for such superior results, see Hildebrand and Tenenbaum [11]. For our purpose here, these superior results on $\Psi(x, y)$ are not needed; the above bounds suffice.

Next consider $P_2(n)$, the second largest prime factor of $n$. Note that whereas $P_1(n)$ is uniquely defined, there are two ways to define the second largest prime factor. We could define $P_2(n) = P_1(n/P_1(n))$ or $P_2(n)$ as the largest prime factor of $n$ strictly less than $P_1(n)$. In the former definition, we set $P_2(n) = 1$ if $\Omega(n) < 2$, and in the latter definition we set $P_2(n) = 1$, if $\omega(n) < 2$. From the point of view of asymptotic estimates, there is little difference between the two ways of defining $P_2(n)$. This is made precise by:

**Theorem 5** *Let $N(x)$ denote the number of positive integers $n \leq x$ for which $P_1(n)$ is a repeated prime factor. Then*

$$N(x) \ll \frac{x}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{(\log x \log\log x)}}}.$$

□

***Proof*** By (3.2b),

$$\Psi(x, e^{\sqrt{(\frac{1}{2}\log x \log\log x)}} << \frac{x}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{(\log x \log\log x)}}}. \tag{3.4}$$

So it suffices to consider those integers $n$ for which $P_1(n) > e^{\sqrt{(\frac{1}{2}\log x \log\log x)}}$. Among these integers $\leq x$, the number of those with largest prime $P_1(n) = p$ repeating, is trivially $O(x/p^2)$. Thus

$$N(x) \ll \Psi\left(x, e^{\sqrt{(\frac{1}{2}\log x \log\log x)}}\right) + \sum_{p > e^{\sqrt{(\frac{1}{2}\log x \log\log x)}}} \frac{x}{p^2}$$

$$\ll \frac{x}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{(\log x \log\log x)}}} + \sum_{n > e^{\sqrt{(\frac{1}{2}\log x \log\log x)}}} \frac{x}{n^2} \ll \frac{x}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{(\log x \log\log x)}}},$$

which proves Theorem 5. □

***Remark*** With more care, the estimate in Theorem 5 can be sharpened, $N(x) = x\exp\{-(\frac{1}{\sqrt{2}} + o(1))(\log x \log\log x)^{1/2}\}$, but the bound in Theorem 5 which was obtained easily, suffices for our purpose.

Here we shall use the definition for $P_2(n)$ as the largest prime factor strictly less than the largest prime factor. From Theorem 5 we see that we can focus on those integers for which $P_1(n)$ occurs square-free.

Consider the counting function

$$\Psi_2(x, y) = \sum_{n \leq x,\ P_2(n) \leq y} 1. \tag{3.5}$$

In contrast to $\Psi(x, y)$ which is very small in comparison with $x$ when $u$ is large, the function $\Psi_2(x, y)$ is not that small. To realize this, observe that all integers of the form $2p \le x$ where $p$ is prime, will have $P_2(n) = 2$, and so

$$\Psi_2(x, y) \ge \Psi_2(x, 2) >> \frac{x}{\log x}, \quad \text{for all} \quad y \ge 2. \tag{3.6}$$

What we need here is a quantitative version of the fact that for "almost all" integers, $P_2(n)$ is large. This (and much more) is provided by a result of Tenenbaum [18], on the size of $P_r(n)$, when $r \ge 2$. We state Tenenbaum's result for $r = 2$ (his eqns (1.5) and (1.6) in [18]) in the form of

**Theorem 6** (Tenenbaum) *There exists a function $\rho_2(u)$ such that*

$$\Psi_2(x, y) = x\rho_2(u)\Big(1 + O\Big(\frac{1}{\log y}\Big)\Big), \quad \textit{uniformly for} \quad 2 \le y \le x. \tag{3.7}$$

*The function $\rho_2(u)$ satisfies*

$$\frac{1}{u} << \rho_2(u) << \frac{1}{u}. \tag{3.8}$$

Tenenbaum's proof of his stronger quantitative result on the joint distribution of the $P_r(n)$ for $r \ge 2$, is quite intricate, and makes use of sharp estimates for $\Psi(x, y)$. He has a precise formula for $\rho_2(u)$ which we do not need here. For our purpose, all we need is

**Corollary** *Uniformly for $2 \le y \le x$, we have*

$$\Psi_2(x, y) \ll \frac{x \log y}{\log x}.$$

□

This follows from Theorem 6. But we point out that the bound in the Corollary for $y \le exp\{(\log x)^{1-\delta}\}$, for any $\delta > 0$, can be proved just by using the bounds in (3.2a) and (3.2b); the implicit constant will depend on $\delta$. The Corollary will be used in what follows.

## 4 The uniform distribution of $P_2(n)$ modulo $k$

In this section, we shall prove:

**Theorem 7** *For each integer $k \ge 2$, the sequence $P_2(n)$ of the second largest prime factors is uniformly distributed in the reduced residue classes modulo k. More precisely, for each fixed $k \ge 2$, and any $1 \le \ell < k$ with $(\ell, k) = 1$, we have*

$$N_2(x, k, \ell) := \sum_{n \le x,\ P_2(n) \equiv \ell (mod\ k)} 1 = \frac{x}{\phi(k)} + O\Big(\frac{x(\log\log x)^2}{\log x}\Big). \tag{4.1}$$

□

***Remark*** Note that the number $n \le x$ with $\omega(n) = 1$ or $\Omega(n) = 1$ is

$$\pi(x) + O(\sqrt{x}) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) = o(x) \tag{4.2}$$

Thus it does not matter in Theorem 7 whether the sum in (4.1) is taken over all integers $n \le x$ for which $P_2(n) \equiv \ell (mod\, k)$, or restricted to integers for which $\omega(n) \ge 2$.

***Proof*** Denote by $S_2(x, p)$ the set of integers $n \le x$ for which $P_2(n) = p$. Then

$$\sum_{p \le \sqrt{x}} |S_2(x, p)| = x - \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{4.3}$$

since the sum accounts for all integers with $\omega(n) \ge 2$. If $N \in S_2(x, p)$, we may write

$$N = mpq, \quad \text{where} \quad q \ge p \quad \text{and} \quad P_1(m) \le p,$$

with $q$ being prime. In particular $m \le (x/p^2)$. Thus

$$\begin{aligned} |S_2(x, p)| &= \sum_{m \le (x/p^2),\, P_1(m) \le p} \ \sum_{q > p,\, mpq \le x} 1 = \sum_{p < q \le x/p} \ \sum_{m \le x/(pq),\, P_1(m) \le p} 1 \\ &= \sum_{p \le q \le x/p} \Psi\left(\frac{x}{pq}, p\right). \end{aligned} \tag{4.4}$$

Thus by summing the expression in (4.4) over $p \le \sqrt{x}$, we get

$$\sum_{p \le \sqrt{x}} |S_2(x, p)| = \sum_{p \le \sqrt{x}} \ \sum_{p \le q \le x/p} \Psi\left(\frac{x}{pq}, p\right) = \sum_{q \le (x/2)} \ \sideset{}{^*}\sum \Psi\left(\frac{x}{pq}, p\right), \tag{4.5}$$

where the * over the inner summation over $p$ on the right means that the conditions on $p$ and $q$ are

$$p \le \sqrt{x}, \quad p \le q, \quad \text{and} \quad p \le \frac{x}{q}. \tag{4.6}$$

To deal efficiently with the double sum on the right in (4.5), we consider two cases, namely $q \le \sqrt{x}$ and $q > \sqrt{x}$, together with the inequalities in (4.6). This gives

$$\sum_{p \le \sqrt{x}} |S_2(x, p)| = \sum_{q \le \sqrt{x}} \ \sum_{p \le q} \Psi\left(\frac{x}{pq}, p\right) + \sum_{\sqrt{x} < q \le (x/2)} \ \sum_{p \le (x/q)} \Psi\left(\frac{x}{pq}, p\right). \tag{4.7}$$

At this point we note that we have an effective version of the statement that $P_2(n)$ is "almost always" large, namely, the Corollary. In view of Corollary, we may consider in (4.7) only the integers $n \le x$ for which $P_2(n) > y$, with $y$ to be chosen later to satisfy $\log y = o(\log x)$. Thus we modify (4.7) to

$$\begin{aligned}\sum := \sum_{y<p\le\sqrt{x}} |S_2(x,p)| &= \sum_{y<q\le\sqrt{x}} \sum_{y<p<q} \Psi\Big(\frac{x}{pq}, p\Big) \\ &+ \sum_{\sqrt{x}<q\le(x/y)} \sum_{y<p\le(x/q)} \Psi\Big(\frac{x}{pq}, p\Big) \\ &= x + O\Big(\frac{x\log y}{\log x}\Big). \end{aligned} \tag{4.8}$$

Next we shall compare $\sum$ with

$$I := \sum_{y<q\le\sqrt{x}} \int_y^q \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log t} + \sum_{\sqrt{x}<q\le(x/y)} \int_y^{x/q} \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log t} \tag{4.9}$$

and estimate the difference (error) $E = \sum - I$ by using the strong form of the PNT. We first consider the absolute value of the difference

$$E_1 := \Big| \sum_{y<q\le\sqrt{x}} \Big\{ \sum_{y<p\le q} \Psi\Big(\frac{x}{pq}, p\Big) - \int_y^q \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log t}\Big\}\Big|$$

$$= \Big| \sum_{y<q\le\sqrt{x}} \Big\{ \sum_{y<p<q} \sum_{n\le(x/pq), P_1(n)\le p} 1 - \int_y^q \Big( \sum_{n\le(x/tq), P_1(n)\le t} 1\Big)\frac{dt}{\log t}\Big\}\Big|$$

$$\le \sum_{y<q\le\sqrt{x}} \sum_{n\le(x/yq)} \Big| \sum_{max(P_1(n),y)\le p\le min(x/nq,q)} 1 - \int_{max(P_1(n),y)}^{min(x/nq,q)} \frac{dt}{\log t}\Big|. \tag{4.10}$$

We now use the strong form of the PNT on the expression on the right in (4.10) to deduce that

$$\begin{aligned} E_1 &\ll \sum_{y<q\le\sqrt{x}} \sum_{n\le(x/yq)} \frac{x}{nq\, exp\{\sqrt{\log(x/nq)}\}} \le \frac{x}{exp(\sqrt{\log y})} \sum_{y<q\le\sqrt{x}} \frac{1}{q} \sum_{n\le(x/yq)} \frac{1}{n} \\ &<< \frac{x\log x}{exp(\sqrt{\log y})} \sum_{y<q\le\sqrt{x}} \frac{1}{q} << \frac{x\log x\log\log x}{exp(\sqrt{\log y})}. \end{aligned} \tag{4.11}$$

In obtaining this upper bound for $E_1$, we have used the fact that $z/exp\sqrt{\log z}$ is an increasing function of $z$, and so in deriving (4.11) we only used the error term in the strong form of the PNT with $z = x/nq$.

Similarly, we bound the difference

$$
\begin{aligned}
E_2 : &= \Big| \sum_{\sqrt{x}<q\le(x/y)} \Big\{ \sum_{y<p<(x/q)} \Psi\Big(\frac{x}{pq}, p\Big) - \int_y^{x/q} \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log t}\Big\}\Big| \\
&= \Big| \sum_{\sqrt{x}<q\le(x/y)} \Big\{ \sum_{y<p<(x/q)} \sum_{n\le(x/pq), P_1(n)\le p} 1 - \int_y^{x/q} \Big( \sum_{n\le(x/tq), P_1(n)\le t} 1\Big)\frac{dt}{\log t}\Big\}\Big| \\
&\le \sum_{\sqrt{x}<q\le(x/y)} \sum_{n\le(x/yq)} \Big| \sum_{max(P_1(n),y)\le p\le(x/nq)} 1 - \int_{max(P_1(n),y)}^{x/nq} \frac{dt}{\log t}\Big| \\
&<< \sum_{\sqrt{x}<q\le(x/y)} \sum_{n\le(x/yq)} \frac{x}{nq\, exp\{\sqrt{\log(x/nq)}\}} \le \frac{x}{exp(\sqrt{\log y})} \sum_{\sqrt{x}<q\le(x/y)} \frac{1}{q} \sum_{n\le(x/yq)} \frac{1}{n} \\
&<< \frac{x\log x}{exp(\sqrt{\log y})} \sum_{\sqrt{x}<q\le(x/y)} \frac{1}{q} << \frac{x\log x\log\log x}{exp(\sqrt{\log y})}. \qquad (4.12)
\end{aligned}
$$

So from (4.11) and (4.12), we see that

$$
|E| = |\sum - I| \le E_1 + E_2 << \frac{x\log x\log\log x}{exp(\sqrt{\log y})}. \qquad (4.13)
$$

From (4.13) and (4.8) we deduce that

$$
I = x + O\Big(\frac{x\log y}{\log x}\Big) + O\Big(\frac{x\log x\log\log x}{exp(\sqrt{\log y})}\Big). \qquad (4.14)
$$

At this stage, we make the choice

$$
y = exp\{(2\log\log x)^2\}, \qquad (4.15)
$$

to conclude that

$$
I = x + O\Big(\frac{x(\log\log x)^2}{\log x}\Big) \quad \text{and} \quad \sum = x + O\Big(\frac{x(\log\log x)^2}{\log x}\Big). \qquad (4.16)
$$

This will be crucial in establishing Theorem 7.

Now for an arbitrary but fixed integer $k \ge 2$, and for any $1 \le \ell < k$ with $(\ell, k) = 1$, we consider the set $S_2^{k,\ell}(x)$ of integers $n \le x$ such that, $\omega(n) \ge 2$, and $P_2(n) \equiv \ell(mod\, k)$. By classifying the members of this set in terms of their second largest prime factor, we see that

$$
N_2(x, k, \ell) = |S_2^{k,\ell}(x)| = \sum_{p\le\sqrt{x},\, p\equiv\ell(mod\, k)} |S_2(x, p)|. \qquad (4.17)
$$

In view of the Corollary, we have

$$
\sum_{p\le y,\, p\equiv\ell(mod\, k)} |S_2(x, p)| \le \sum_{p\le y} |S_2(x, p)| << \frac{x\log y}{\log x}. \qquad (4.18)
$$

Thus

$$N_2(x,k,\ell) = |S_2^{k,\ell}(x)| = \sum_{y<p\le\sqrt{x},\ p\equiv\ell(mod\ k)} |S_2(x,p)| + O\Big(\frac{x\log\ y}{\log\ x}\Big). \tag{4.19}$$

Let us denote the sum on the right of (4.19) as $\sum^{k,\ell}$. Then by reasoning as above, we get

$$\sum^{k,\ell} = \sum_{y<q\le\sqrt{x}}\ \sum_{y<p<q,\ p\equiv\ell(mod\ k)} \Psi\Big(\frac{x}{pq}, p\Big) + \sum_{\sqrt{x}<q\le(x/y)}\ \sum_{y<p\le(x/q),\ p\equiv\ell(mod\ k)} \Psi\Big(\frac{x}{pq}, p\Big). \tag{4.20}$$

We now want to compare the expression in (4.20) with

$$I^{k,\ell} := \sum_{y<q\le\sqrt{x}} \frac{1}{\phi(k)}\int_y^q \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log\ t} + \sum_{\sqrt{x}<q\le(x/y)} \frac{1}{\phi(k)}\int_y^{x/q} \Psi\Big(\frac{x}{tq}, t\Big)\frac{dt}{\log\ t}. \tag{4.21}$$

Clearly

$$I^{k,\ell} = \frac{I}{\phi(k)}, \tag{4.22}$$

with $I$ as in (4.9). Just as we obtained the bound in (4.13) for the difference $\sum - I$ using the strong form of the PNT, we can use the same reasoning together with the strong form of the PNT for arithmetic progressions to deduce that

$$\sum^{k,\ell}\ -\ I^{k,\ell} << \frac{x\log\ x\log\log\ x}{exp(\sqrt{\log\ y})}. \tag{4.23}$$

So from the above estimates, we deduce that

$$N_2(x,k,\ell) = |S_2^{k,\ell}(x)| = \frac{x}{\phi(k)} + O\Big(\frac{x\log\ y}{\log\ x}\Big) + O\Big(\frac{x\log\ x\log\log\ x}{exp(\sqrt{\log\ y})}\Big). \tag{4.24}$$

Once again,we make the choice

$$y = exp\{(2\log\log\ x)^2\}, \tag{4.25}$$

to deduce Theorem 7 from (4.24). □

## 5 Proof of the main result

Theorem 7 paves the way to the proof of our main result (Theorem 10 below). Enroute to Theorem 10, we establish two theorems, the first of which relies on Theorem 7:

**Theorem 8** *For integers $\ell, k$ satisfying $1 \leq \ell \leq k$ with $(\ell, k) = 1$, we have*

$$M_\omega(x; \ell, k) := \sum_{n \leq x,\ p_1(n) \equiv \ell (mod\, k)} \mu(n)\omega(n) \ll \frac{x(\log\log\ x)^4}{\log\ x}.$$

□

***Proof*** Let $f$ be a function on the primes defined by

$$f(p) = 1 \quad \text{if} \quad p \equiv \ell (mod\, k), \quad f(p) = 0, \quad \text{otherwise.} \tag{5.1}$$

Then by (1.13) and Moebius inversion, we get

$$\sum_{1<n \leq x,\ p_1(n) \equiv \ell (mod\, k)} \mu(n)(\omega(n) - 1) = \sum_{1<n \leq x} \mu(n)(\omega(n) - 1) f(p_1(n))$$

$$= \sum_{1<n \leq x} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) f(P_2(d),$$

which we rewrite as

$$\sum_{1<n \leq x, p_1(n) \equiv \ell (mod\, k)} \mu(n)\omega(n) = \sum_{1<n \leq x} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) f(P_2(d) + \sum_{1<n \leq x,\ p_1(n) \equiv \ell (mod\, k)} \mu(n)$$

$$:= \Sigma_5 + \Sigma_6 \quad \text{respectively} \tag{5.2}$$

It was already established in [2] that

$$\Sigma_6 << x\ \exp\{-(\log\ x)^{(1/3)}\}. \tag{5.3}$$

With regard to $\Sigma_5$, we employ the hyperbola method and write it as

$$\Sigma_5 = \sum_{m \leq T} \mu(m) \sum_{d \leq (x/m)} f(P_2(d)) + \sum_{d \leq (x/T)} f(P_2(d)) \sum_{T \leq m \leq (x/d)} \mu(m)$$

$$:= \Sigma_7 + \Sigma_8, \quad \text{respectively.} \tag{5.4}$$

Using (2.2) we get

$$\Sigma_8 << \sum_{d \leq (x/T)} f(P_2(d)) \frac{x}{d\ \exp(c_1\sqrt{\log(x/d)})} << \frac{x\ \log\ x}{\exp(c_1\sqrt{\log\ T})}. \tag{5.5}$$

Regarding $\Sigma_7$, Theorem 7 gives

$$\Sigma_7 = \sum_{m \le T} \mu(m)\Big\{\Big(\frac{x}{\phi(k)m}\Big) + O\Big(\frac{x(\log\log x)^2}{m\log(x/m)}\Big)\Big\}$$
$$= \frac{x}{\phi(k)} \sum_{m \le T} \frac{\mu(m)}{m} + O\Big(\frac{x\log T(\log\log x)^2}{\log(x/T)}\Big). \tag{5.6}$$

We now use the bound

$$\sum_{m \le T} \frac{\mu(m)}{m} \ll \frac{1}{\exp(c_1\sqrt{\log T})}$$

which follows from the method that gives (2.2), and this along with (5.6) yields

$$\Sigma_7 << \frac{x}{\phi(k)\exp\{c_1\sqrt{(\log T)}\}} + \frac{x\log T(\log\log x)^2}{\log(x/T)}. \tag{5.7}$$

At this point we choose

$$T = \exp\{4c_1^{-2}(\log\log x)^2\} \quad <=> \quad \sqrt{\log T} = 2c_1^{-1}\log\log x.$$

With this choice of $T$, we deduce from (5.2)–(5.7) that

$$\sum_{n \le x,\ p(n)\equiv \ell (mod\, k)} \mu(n)\omega(n) \ll \frac{x(\log\log x)^4}{\log x}$$

which proves Theorem 8. □

We next prove

**Theorem 9** *Let $\ell, k$ be integers satisfying $1 \le \ell \le k$ with $(\ell, k) = 1$. Then*

$$\sum_{1<n \le x,\ p_1(n)\equiv \ell (mod\, k)} \mu(n)\omega(n)\Big\{\frac{x}{n}\Big\} \ll \frac{x(\log\log x)^{5/2}}{\sqrt{\log x}},$$

*where $\{y\}$ denotes the fractional part of $y$.* □

***Proof*** To prove Theorem 9, we choose

$$a_n = \mu(n)\omega(n)f(p(n)) \tag{5.8}$$

in Theorem A, where $f$ is as in (5.1). For this choice, we see that Theorem 8 shows that we can take

$$\eta(x) = \frac{(\log\log x)^4}{\log x}, \tag{5.9}$$

in (2.20). Thus (2.21a) and (2.21b) hold. Since $|a_n| \le \omega(n)$, we can take

$$\beta(x) = \log\log x \tag{5.10}$$

in (2.22). Thus (2.23) holds. So by (5.9), (5.10), and (2.24), we get Theorem 9. □

We are now in a position to prove our main result:

**Theorem 10** *For integers $\ell, k$ satisfying $1 \le \ell \le k$ and $(\ell, k) = 1$, we have*

$$m_\omega(x;\ell,k) := \sum_{n\le x,\ p_1(n)\equiv\ell(\bmod k)} \frac{\mu(n)\omega(n)}{n} \ll \frac{(\log\log x)^{5/2}}{\sqrt{\log x}}.$$

*By letting $x \to \infty$ in the preceding estimate, we get*

$$\sum_{n\ge 2,\ p_1(n)\equiv\ell(\bmod k)} \frac{\mu(n)\omega(n)}{n} = 0.$$

□

***Proof*** With $f(p)$ defined on the primes as above, note that

$$\sum_{1<d\le x} \mu(d)(\omega(d)-1)f(p_1(d))\Big[\frac{x}{d}\Big] = \sum_{n\le x}\sum_{1<d|n} \mu(d)(\omega(d)-1)f(p_1(d))$$

$$= \sum_{n\le x} f(P_2(n)) = \frac{x}{\phi(k)} + O\Big(\frac{x(\log\log x)^2}{\log x}\Big) \tag{5.11}$$

by (1.13) and Theorem 7.

It was already established in [2] that

$$\sum_{1<n\le x} \mu(d)f(p_1(d))\Big[\frac{x}{d}\Big] = \sum_{n\le x}\sum_{1<d|n} \mu(d)f(p_1(d))$$

$$= -\sum_{n\le x} f(P_1(n)) = \frac{-x}{\phi(k)} + O\Big(\frac{x}{\exp\{(\log x)^{1/3}\}}\Big). \tag{5.12}$$

On comparing (5.11) and (5.12), we see that the main term $x/\phi(k)$ cancels, and so

$$\sum_{1<d\le x} \mu(d)\omega(d)f(p_1(d))\Big[\frac{x}{d}\Big] = O\Big(\frac{x(\log\log x)^2}{\log x}\Big). \tag{5.13}$$

But we know by Theorem 9 that

$$\sum_{1<n\leq x} \mu(d)\omega(d)f(p_1(d))\{\frac{x}{d}\} = O\Big(\frac{x(\log\log x)^{5/2}}{\sqrt{\log x}}\Big). \tag{5.14}$$

Finally by adding the expressions in (5.13) and (5.14), we get

$$x\sum_{1<n\leq x} \frac{\mu(d)\omega(d)f(p_1(d))}{d} = O\Big(\frac{x(\log\log x)^{5/2}}{\sqrt{\log x}}\Big). \tag{5.15}$$

On dividing both sides of (5.15) by $x$, we get Theorem 10. □

## 6 Sums involving the exceptional primes

In Sect. 2 we proved (Theorem 4) that

$$\sum_{n=2}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0. \tag{6.1}$$

Then we proved in Sect. 5 that if $k \geq 2$ is an arbitrary modulus, then for every $\ell$ that satisfies $(\ell, k) = 1$

$$\sum_{n=2,\ p_1(n)\equiv\ell(mod\, k)}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0. \tag{6.2}$$

When we sum the expression on the left in (6.2) over all $1 \leq \ell < k$ with $(\ell, k) = 1$, we do not get the full sum in (7.1) because the primes

$$p \equiv \ell(mod\, k) \quad \text{with} \quad (\ell, k) > 1, \tag{6.3}$$

have not been accounted for. But there will be primes satisfying the conditions in (6.3), which we call *exceptional primes*, if and only if $\ell$ is a prime divisor of $k$, and in this case there is just a single prime $p$ in the residue class $\ell(mod\, k)$, namely $p = \ell$. It turns out that the sum in (6.2) is 0 when taken over $n$ satisfying $p(n) = p$ for any fixed prime regardless of whether $p$ divides $k$ or not. That is we have

**Theorem 11** *Let $p$ be an arbitrary but fixed prime. Then*

$$\sum_{n=1,\ p_1(n)=p}^{\infty} \frac{\mu(n)\omega(n)}{n} = \sum_{n=2,\ p_1(n)=p}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0.$$

□

***Proof*** The square-free integers $n$ with $p(n) = p$ are those of the form

$$n = mp, \quad \text{with} \quad (m, N_p) = 1, \quad \text{where} \quad N_p = \prod_{q \le p,\, q = prime} q. \tag{6.4}$$

Thus using $\omega(mp) = \omega(m) + 1$, we get

$$\begin{aligned}\sum_{n=2,\, p_1(n)=p}^{\infty} \frac{\mu(n)\omega(n)}{n} &= \frac{-1}{p} \sum_{(m,N_p)=1} \frac{\mu(m)\omega(mp)}{m} \\ &= -\frac{1}{p} \sum_{(m,N_p)=1} \frac{\mu(m)}{m} - \frac{1}{p} \sum_{(m,N_p)=1} \frac{\mu(m)\omega(m)}{m} \\ &= \Sigma_9 + \Sigma_{10}.\end{aligned}$$

It is a classical result the $\Sigma_9 = 0$. The methods of Sect. 2 can be used to show that $\Sigma_{10} = 0$. Thus Theorem 11 follows from (6.5). □

***Remark*** (i) Since the exceptional primes, namely those that divide the modulus $k$, are finite in number, the sum of the expression in Theorem 11 taken over all exceptional primes is 0 since it is a sum of a finite number of zeros. Thus by Theorem 11, the exceptional primes are accounted for in the full sum in (6.1). When $(\ell, k) = 1$, there are infinitely many primes $p \equiv \ell (mod\, k)$, and for each $p$ in the residue class $\ell (mod\, k)$, the sum as in Theorem 11 is 0. What makes Theorem 10 interesting is that we are summing "infinitely many zeros", yet the sum is 0.

(ii) As was the case with our earlier theorems, a quantitative version of Theorem 11 can be established.

## 7 The case of general *f*

In the penultimate section of [2], it was shown that if $f$ is ANY bounded function on the primes, then

$$M_f(x) := \sum_{2 \le n \le x} \mu(n) f(p_1(n)) = o(x). \tag{7.1}$$

From (8.1), it follows by Axer's theorem that

$$\sum_{2 \le n \le x} \mu(n) f(p_1(n)) \{\frac{x}{n}\} = o(x), \tag{7.2}$$

where $\{t\}$ denotes the fractional part of $t$. Next, by the Duality identity (1.3) we have

$$\sum_{2 \le n \le x} \mu(n) f(p_1(n)) [\frac{x}{n}] = - \sum_{2 \le n \le x} f(P_1(n)). \tag{7.3}$$

Hence by adding the expressions in (7.2) and (7.3), we get

$$x \sum_{2 \le n \le x} \frac{\mu(n) f(p_1(n)}{n} = - \sum_{2 \le n \le x} f(P_1(n)) + o(x). \tag{7.4}$$

From (7.4), the equivalence of (1.5) and (1.6) follows, and this was how this equivalence was proved in [2].

In [2], the following simple bound

$$M_f(x) << \frac{x}{\log \log \log\, x} \tag{7.5}$$

was established, but subsequently in [3] it was refined to

$$\max_{|f| \le 1|} |M_f(x)| \sim \frac{2x}{\log\, x}. \tag{7.6}$$

Of course, for specific functions $f$, such as $f$ being the characteristic function of primes in an arithmetic progression $\ell(mod\, k)$, where $(\ell, k) = 1$, the bound for $M_f(x)$ is vastly superior (see [2]).

Similar in spirit to (7.1), it can be shown that

**Theorem 12** *If $f$ is any bounded function on the primes, then*

$$M_{f,\omega}(x) := \sum_{n \le x} \mu(n)\omega(n) f(p_1(n)) = o(x).$$

□

A proof of a quantitative version of Theorem 12 will be given in Alamoudi–Alladi [1]. From Theorem 12 and Theorem A, it will follow that

$$\sum_{n \le} \mu(n)\omega(n) f(p_1(n)) \{\frac{x}{n}\} = o(x). \tag{7.7}$$

While all this seems to be similar to (7.1) and (7.2), an important difference occurs here. In order to apply the Duality identity (1.13), we have to consider the sum

$$\begin{aligned} \sum_{2 \le n \le x} \mu(n)(\omega(n) - 1) f(p_1(n))[\frac{x}{n}] &= \sum_{2 \le n \le x} \sum_{1 < d | n} \mu(d)(\omega(d) - 1) f(p_1(d) \\ &= \sum_{2 \le n \le x} f(P_2(n)). \end{aligned} \tag{7.8}$$

We note that the first sum on the left hand side of (7.8) is

$$\sum_{2 \le n \le x} \mu(n)\omega(n) f(p_1(n))[\frac{x}{n}] - \sum_{2 \le n \le x} \mu(n) f(_1(n))[\frac{x}{n}]$$

$$= \sum_{2 \le n \le x} \mu(n)\omega(n) f(p_1(n))\Big[\frac{x}{n}\Big] + \sum_{2 \le n \le x} f(P_1(n)) \tag{7.9}$$

in view of (7.3). So from (7.7), (7.8), and (7.9), we get

$$x \sum_{2 \le n \le x} \frac{\mu(n)\omega(n) f(p_1(n))}{n} = \sum_{2 \le n \le x} f(P_2(n)) - \sum_{2 \le n \le x} f(P_1(n)). \tag{7.10}$$

Now (7.10) yields the following result:

**Theorem 13** *If $f$ is a bounded function on the primes such that*

$$\sum_{2 \le n \le x} f(P_1(n)) \sim \kappa x \tag{7.11}$$

*and*

$$\sum_{2 \le n \le x} f(P_2(n)) \sim \kappa x, \tag{7.12}$$

*for some constant $\kappa$, then*

$$\sum_{n=2}^{\infty} \frac{\mu(n)\omega(n) f(p_1(n))}{n} = 0. \tag{7.13}$$

□

***Remark*** When $f(p)$ defined on primes $p$ is the characteristic function of primes in the residue class $\ell(mod\, k)$, where $(\ell, k) = 1$, then $\kappa = 1/\phi(k)$ in (7.11) and (7.12), in which case (7.13) is Theorem 10. If instead of the same contant $\kappa$ in (7.12) and (7.13), we had two different constants, $\kappa_1$ in (7.11) and $\kappa_2$ in (7.12), then the sum in (7.13) will converge to $\kappa_2 - \kappa_1$. But we wish to stress that we know of no natural example of a bounded function on the primes for which the constants $\kappa_1$ and $\kappa_2$ have different values. Thus we pose

PROBLEM: *Does there exist a bounded function f on the primes such that*

$$\sum_{2 \le n \le x} f(P_1(n)) \sim \kappa_1 x, \quad \text{and} \quad \sum_{2 \le n \le x} f(P_2(n)) \sim \kappa_2 x, \quad \text{with} \quad \kappa_1 \ne \kappa_2?$$

Consider now the following situation. Given $x$ arbitrarily large, define a function $f$ on the primes as follows:

$$f(p) = 1 \text{ if } \sqrt{x} < p \le x, \quad f(p) = 0 \text{ if } p \le \sqrt{x}. \tag{7.14}$$

With $f$ as in (7.14), we have

$$\sum_{2 \le n \le x} f(P_1(n)) = \sum_{\sqrt{x} < p \le x} \sum_{n \le x,\, P_1(n) = p} 1$$

$$\sum_{\sqrt{x}<p\leq x}\left[\frac{x}{p}\right]=x log\,2+O\Big(\frac{x}{log\,x}\Big). \tag{7.15}$$

On the other hand, since $P_2(n)\leq\sqrt{x}$ if $n\leq x$, we clearly have

$$\sum_{2\leq n\leq x} f(P_2(n))=0.$$

So in this example, $\kappa_1=log\,2$, and $\kappa_2=0$. But note that the definition of $f$ in (7.14) depends on $x$, whereas in Problem 1 we ask for a function $f$ just defined on the primes (without dependency on $x$).

The importance of the consideration of general functions $f$ in this section will be clear in the next section when we will discuss algebraic analogues to the results of Alladi [2] by various authors, and algebraic analogues of the results in this paper by Sengupta [16].

## 8 Algebraic and *q*-analogues, and higher order duality

The Duality identity (1.3), and the result (1.8) established in quantitative form in Alladi [2], have been investigated and generalized in an algebraic setting by various authors in the last decade. It all started with the paper [6] of Dawsey who obtained the following algebraic analogue and extension of (1.8) to Galois extensions of the field of rationals $\mathbb{Q}$:

Let $K$ be a Galois extension of $\mathbb{Q}$, and $\mathcal{O}_K$ the ring of integers in $K$. If $p$ is a prime in the integers, then let $P$ denote the prime ideal that is contained in $\mathcal{O}_K$ which lies above $p$. If $p$ is unramified, let $[\frac{K/\mathbb{Q}}{P}]$ denote the Artin symbol. For simplicity, let

$$\left[\frac{K/\mathbb{Q}}{p}\right]:=\left[\frac{K/\mathbb{Q}}{P}\right].$$

Then

**Theorem D** (Dawsey) *Let $K$ be a finite Galois extension of $\mathbb{Q}$ with Galois group $G=Gal(K/\mathbb{Q})$. Let $C$ be a conjugacy class in $G$. Then*

$$-\sum_{n\geq 2,\left[\frac{K/\mathbb{Q}}{p_1(n)}\right]=C}\frac{\mu(n)}{n}=\frac{|C|}{|G|}.$$

□

Dawsey notes that Theorem D is a generalization of (1.8), because in the special case when $K$ is a cyclotomic extension of $\mathbb{Q}$, the group $Gal(K/\mathbb{Q})$ can be identified with $\mathbb{Z}_k^*$, the set of reduced residues modulo $k$ for some positive integer $k$; since $\mathbb{Z}_k^*$ is

Abelian, each conjugacy class has just one element and so

$$\frac{|C|}{|G|} = \frac{1}{\phi(k)}.$$

Dawsey uses the Chebotarev Density Theorem to show that

$$\sum_{2 \le n \le x, \left[\frac{K/\mathbb{Q}}{P_1(n)}\right]=C} 1 \sim \frac{|C|}{|G|} x. \tag{8.1}$$

Then by the methods in [2] that involve Duality, Dawsey is able to get Theorem D.

Motivated by Dawsey's work, Sweeting and Woo [17] obtained a generalization of Theorem D in which the finite extensions $K$ of $\mathbb{Q}$ are replaced by finite extensions $L$ of an arbitrary algebraic number field $K$. In discussing this more general situation, Sweeting and Woo consider a generalization of the Moebius function defined in terms of products of prime ideals instead of product of primes, and establish a duality identity that generalizes (1.3) appropriately. In this more general situation, the Chebotarev Density Theorem applies, and so an analogue of Theorem D is established in [17].

While it is true that Theorem D generalizes (1.8), it is to be noted that the more general equivalence of (1.5) and (1.6) is established as Theorem 6 in [2]. So what Dawsey confirmed is that if $f$ is chosen to be the characteristic function of primes $p$ for which the Artin symbol $[\frac{K/\mathbb{Q}}{P}] = C$, then the average of $f(P_1(n))$ exists. That is, in this case $c$ in (1.5) is $|C|/|G|$. So the deduction of Theorem D from (9.1) is a special case of the equivalence of (1.5) and (1.6). Since the equivalence of (1.5) and (1.6) is established in [2] for arbitrary bounded functions $f$, the bound for the quantitative version of (1.6) is weak. For the Chebotarev Density Theorem, Lagarias and Odlyzko [13] have established a strong form, with the error term comparable to the error term in the strong form of the PNT. Thus utilizing the Lagarias–Odlyzko theorem, Dawsey is able to get a superior quantitative version of Theorem D where the bound is just as sharp as the quantitative version of (1.8) that is proved in [2] using the strong form of the PNT.

The results of Sweeting and Woo have been extended by Kural et al. [12]. A generalization in a different direction, namely replacing the Moebius function by the more general Ramanujan sum

$$c_m(n) = \sum_{k=1,(k,n)=1}^{n} e^{2imk\pi/n}, \tag{8.2}$$

is considered by Wang [22] ($\mu(n) = c_1(n)$). Also, Wang in collaboration with Duan and Yi [9] has discussed analogues of Alladi's duality in global function fields. Motivated by the results of this paper involving $\mu(n)\omega(n)$, with $\omega(n)$ being an additive function, Wang [24] has very recently discussed the consequences of the logarithmic analog of the Duality (1.3), by replacing $\mu(n)$ with $\mu(n) \log n$.

A fruitful way to generalize arithmetic results is to obtain suitable $q$-analogues. In two papers [14, 15], Ono–Schneider–Wagner have discussed a variety of $q$-analogues of arithmetic density results and their partition implications.

With regard to the arithmetic consequences of the second order duality (namely consequences of (1.13)), recently Sengupta [16], motivated by the work of Dawsey [6], has obtained the extension of Theorem 10 to the situation when $K$ is a finite Galois extension of $\mathbb{Q}$. Like Dawsey, Sengupta uses the strong form of the Chebotarev Density Theorem due to Lagarias and Odlyzko [13].

We mention that Alladi and Sengupta [4] have very recently considered arithmetic consequences of higher order dualities, namely (1.12) for $r \geq 3$, and established analogues of all the results in this paper for $r \geq 3$. In this discussion of higher order dualities, it turns out that when $r \geq 3$, the bounds for certain terms have extra factors which are powers of $\log u$, where $u = \log x / \log y$; these factors are not present in the case $r = 2$ treated here.

Finally, we point out that all the quantitative results in [2] were established with uniformity for the moduli $k$ of arithmetic progressions satisfying $k \leq \log^{\beta} x$, with implicit contants depending on $\beta$. This is because in [2], we utilized the Siegel-Walfisz theorem for primes in arithmetic progressions. If we had used the Siegel-Walfisz theorem here, then Theorem 10 would hold with uniformity for $k \leq \log^{\beta} x$.

## 9 Concluding remarks

(i) *Arithmetic density versions:*

The generalizations of (1.8) to algebraic number fields by various authors starting with Dawsey [6] was motivated by rewriting (1.8) as

$$-\sum_{n \geq 2,\ p_1(n) \equiv \ell (mod\, k)} \frac{\mu(n)}{n} = \frac{1}{\phi(k)}, \tag{9.1}$$

and interpreting this as an arithmetic density result. Similarly, our Theorem 10 can be rewritten as

$$\sum_{n \geq 2, p_1(n) \equiv \ell (mod\, k)} \frac{\mu(n)(\omega(n)-1)}{n} = \frac{1}{\phi(k)}, \tag{9.2}$$

and interpreted as an arithmetic density result, thereby lending itself to an arithmetic density generalization to algebraic number fields using the Chebotarev density theorem (see Sengupta [16]). The consequence of the general identity (1.12) for $r \geq 3$ discussed in Alladi–Sengupta [4] also has an arithmetic density formulation, namely

$$(-1)^r \sum_{n \geq 2, !(n) \equiv \ell (mod\, k)} \frac{\mu(n)}{n} \binom{\omega(n)-1}{r-1} = \frac{1}{\phi(k)}. \tag{9.3}$$

This can be generalized to algebraic number fields using the Chebotarev density theorem.

*Tenenbaum's generalization of Theorem 10:*

Very recently, Tenenbaum [21] has generalized Theorem 10 as follows:

**Theorem T** *Let* $\mathbb{P}$ *be a set of primes satisfying*

$$\varepsilon(t) = \frac{1}{t}\Big\{ \sum_{p \le t,\, p \in \mathbb{P}} \log p \Big\} - \kappa = o(1), \quad as \quad t \to \infty, \tag{9.4}$$

*with some* $\kappa \in [0, 1]$. *Then*

$$\sum_{n \ge 2,\, p_1(n) \in \mathbb{P}}^{\infty} \frac{\mu(n)\omega(n)}{n} = 0. \tag{9.5}$$

□

Tenenbaum's proof of a quantitative form of (9.5) is analytic and quite intricate. But the main thing is that he is able to get (9.5) directly from (9.4) without relying on estimates like (7.11) and (7.12). But then, our approach using Duality connecting sums involving $\mu(n)\omega(n) f(p_1(n))$ with $f(P_1(n))$ and $f(P_2(n))$ is of intrinsic interest, and that is the motivation of the present paper.

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