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## 2 Some $q$ -hypergeometric identities associated with partition 3 theorems of Lebesgue, Schur and Capparelli

Dedicated to Mourad Ismail on the occasion of his 80th birthday

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5 Received: 9 February 2025 / Accepted: 5 October 2025  
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7 **Abstract** Here, we establish a polynomial identity in three variables  $a, b, c$ , and with the degree of the  
8 polynomial given in terms of two integers  $L, M$ . By letting  $L$  and  $M$  tend to infinity, we get the 1993 Alladi–  
9 Gordon  $q$ -hypergeometric key-identity for the generalized Schur Theorem as well as the fundamental Lebesgue  
10 identity by two different choices of the variables. This polynomial identity provides a generalization and a  
11 unified approach to the Schur and Lebesgue theorems. We discuss other analytic identities for the Lebesgue  
12 and Schur theorems and also provide a key identity ( $q$ -hypergeometric) for Andrews’ deep refinement of the  
13 Alladi–Schur theorem. Finally, we discuss a new infinite hierarchy of identities, the first three of which relate  
14 to the partition theorems of Euler, Lebesgue, and Capparelli, and provide their polynomial versions as well.

15 **Mathematics Subject Classification** 05A15 · 05A17 · 11P81 · 11P83

### 16 1 Introduction

17 One of the fundamental  $q$ -hypergeometric identities is Lebesgue’s identity:

$$18 \sum_{i=0}^{\infty} \frac{q^{T_i} (-cq)_i}{(q)_i} = \prod_{m=1}^{\infty} (1+q^m)(1+cq^{2m}) = \prod_{m=1}^{\infty} \frac{(1+cq^{2m})}{(1-q^{2m-1})}. \quad (1.1)$$

20 The importance of (1.1) is due to the fact that when  $c = 0$  it yields Euler’s series and product generating  
21 functions for partitions into distinct parts, and with the dilations and translations given by

$$22 \text{dilation } q \mapsto q^2, \quad \text{translations } c \mapsto cq^{-1} \text{ or } c \mapsto cq, \quad (1.2)$$

24 it yields the  $q$ -hypergeometric identities for the Little Göllnitz partition theorems (see Theorem G below).  
25 In [8], Alladi–Gordon gave the following combinatorial interpretation of Lebesgue’s identity as a weighted  
26 partition theorem along with a combinatorial proof:

27 **Theorem L** *Let  $D(n; j)$  denote the number of partitions of  $\pi : b_1 + b_2 + \dots + b_v = n$  into distinct parts  $b_i$ ,  
28 such that there are  $j$  gaps  $b_i - b_{i+1} \geq 2$  among the parts for  $i = 1, 2, \dots, v$ , with the convention  $b_{v+1} = 0$ .*

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29 Let  $C(n; k)$  denote the number of partitions of  $n$  with even parts non-repeating, such that there are precisely  
 30  $k$  even parts. Then

$$31 \quad \sum_j D(n, j)(1+c)^j = \sum_k C(n; k)c^k.$$

33 Under the transformations in (1.2), the two Göllnitz identities that emerge are:

$$34 \quad \sum_{i=0}^{\infty} \frac{q^{i^2+i}(-cq^{-1}; q^2)_i}{(q^2; q^2)_i} = \prod_{m=1}^{\infty} (1+q^{4m})(1+q^{4m-2})(1+cq^{4m-3}), \quad (1.2a)$$

36 and

$$37 \quad \sum_{i=0}^{\infty} \frac{q^{i^2+i}(-cq; q^2)_i}{(q^2; q^2)_i} = \prod_{m=1}^{\infty} (1+q^{4m})(1+q^{4m-2})(1+cq^{4m-1}). \quad (1.2b)$$

39 The Little Göllnitz theorem(s) [26], which are the partition interpretations of (1.2a) and (1.2b), are:

40 **Theorem G** For  $i = 1, 2$ , let  $g_i(n; k)$  denote the number of partitions of  $n$  into parts that differ by  $\geq 2$ , with  
 41 strict inequality if a part is odd, having  $k$  odd parts, where the smallest part is  $\geq i$ .

42 For  $i = 1, 2$ , let  $G_i(n; k)$  denote the number of partitions into distinct parts, which are of the form  
 43  $2, 4$ , or  $2i - 1 \pmod{4}$ , and with  $k$  parts  $\equiv 2i - 1 \pmod{4}$ . Then

$$45 \quad g_i(n; k) = G_i(n; k), \quad \text{for } i = 1, 2.$$

46 **Remark 1.1** It is only at the undilated level, that is, for Lebesgue's identity, the partition theorem (Theorem  
 47 L) is a weighted partition theorem. Once we have the dilation as in (1.2), Theorem G is a regular partition  
 48 theorem (not weighted).

49 The celebrated 1926 partition theorem of Schur is:

50 **Theorem S** Let  $B(n)$  denote the number of partitions of  $n$  into distinct parts  $\equiv \pm 1 \pmod{3}$ .

51 Let  $S(n)$  denote the number of partitions of  $n$  into parts that differ by at least 3, with strict inequality if a  
 52 part is a multiple of 3. Then

$$53 \quad S(n) = B(n).$$

55 **Remark 1.2** Note one similarity in the difference conditions in Theorems G and S, namely, in Theorem G, the  
 56 gap between parts is  $\geq 2$  with strict inequality if a part is odd, while in Theorem S, the gap between parts is  
 57  $\geq 3$  with strict inequality if a part is a multiple of 3. For the remainder of this manuscript, we will refer to the  
 58 partitions enumerated by  $S(n)$  as *Schur partitions*.

59 Gleissburg [24] showed that Theorem S can be refined to

$$60 \quad B(n; k) = S(n; k),$$

62 where  $B(n; k)$  and  $S(n; k)$  denote the number of partitions enumerated by  $B(n)$  and  $S(n)$  with the condition  
 63 that the number of parts is  $k$ , and with the convention that parts which are multiples of 3 are counted twice by  
 64  $S(n; k)$ .

65 In 1993, Alladi and Gordon [9] proved a two-parameter refinement and generalization of Theorem S, and  
 66 in doing so, for the first time, cast Theorem S in the form of a  $q$ -hypergeometric identity, which they dubbed  
 67 a *key-identity*:

$$68 \quad \sum_{\alpha, \beta, \gamma} \frac{a^{\alpha+\gamma} b^{\beta+\gamma} q^{T_s+T_\gamma}}{(q)_\alpha (q)_\beta (q)_\gamma} = \sum_{i, j} \frac{a^i b^j q^{T_i+T_j}}{(q)_i (q)_j} = \prod_{i=1}^{\infty} (1+aq^i)(1+bq^i), \quad (1.3)$$

70 where  $s = \alpha + \beta + \gamma$ .

71 By using the transformations

$$72 \quad (\text{dilation}) q \mapsto q^3, \quad \text{and} \quad (\text{translations}) a \mapsto aq^{-2}, b \mapsto bq^{-1} \quad (1.4)$$

74 in (1.3), the following strong refinement of Theorem S falls out:



75 **Theorem A-G** Let  $B(n; i, j)$  denote the number of partitions of  $n$  into  $i$  distinct parts  $\equiv 1 \pmod{3}$ , and  $j$   
 76 distinct parts  $\equiv 2 \pmod{3}$ .

77 Let  $S(n; \alpha, \beta, \gamma)$  denote the number of partitions of the type enumerated by  $S(n)$ , with the condition that  
 78 the number of parts  $\equiv 1, 2, 3 \pmod{3}$  is  $\alpha, \beta, \gamma$  respectively. Then

$$79 \quad \sum_{\alpha+\gamma=i, \beta+\gamma=j} S(n; \alpha, \beta, \gamma) = B(n; i, j).$$

80 Notice that in Theorem A-G, the total number of parts is

$$82 \quad i + j = \alpha + \gamma + \beta + \gamma = \alpha + \beta + 2\gamma,$$

83 and so the parts that are multiples of 3 are counted twice. In [9], the combinatorial interpretation of the key-  
 84 identity was given in terms of partitions into parts occurring in three possible colors: two primary colors,  $a$   
 85 and  $b$ , and one secondary color,  $ab$  the combination of the other two, with gap conditions on the colored parts,  
 86 and with the condition that the parts occurring in the secondary color are counted twice. Under the dilation  
 87 and translations given in (1.4), the parts in primary colors  $a, b$  correspond to parts  $\equiv 1, 2 \pmod{3}$ , and parts  
 88 in secondary color are then the multiples of 3.

89 The colored partition version (generalization) of Theorem A-G is proved combinatorially (bijectively) in  
 90 [9]. The combinatorial proofs of the weighted partition Theorem L given in [8], and of the colored partition  
 91 version of Theorem A-G in [9] are similar, with the main difference being in the final step, where in the case  
 92 of Theorem L, a certain choice could be made; this is why  $D(n; j)$  has a weight  $(1+c)^j$  attached to it. Since  
 93 the combinatorial proofs of Theorem L and the colored generalization of Theorem A-G are so similar, it is  
 94 natural to ask if there is a unified  $q$ -hypergeometric approach to Lebesgue's identity (1.1) and the key-identity  
 95 (1.3) for the generalized Schur theorem? After establishing a finite analog of Lebesgue's identity in Sect. 3,  
 96 we provide in Sect. 6 a new polynomial identity, from which, under two different specializations, the finite  
 97 Schur and Lebesgue identities fall out. Following this, in Sect. 6, we provide a  $q$ -hypergeometric *key-identity*  
 98 for Andrews' deep refinement of the Alladi–Schur Theorem. Finally, in Sect. 7, we introduce a new infinite  
 99 hierarchy of identities of which the first three correspond to the partition theorems of Euler, Lebesgue, and  
 100 Capparelli; we provide a polynomial version of this infinite hierarchy as well.

101 To aid the reader, we will conclude this introductory section by recalling basic facts, along with notational  
 102 conventions, that are used throughout the paper.

103 For complex numbers  $a, q$ , we use the  $q$ -Pochhammer symbols

$$105 \quad (a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j),$$

106 and

$$108 \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n = \prod_{j=0}^{\infty} (1 - aq^j), \quad \text{if } |q| < 1.$$

110 The variable  $q$  is called the base. We often write  $(a)_n$  in place of  $(a; q)_n$  suppressing  $q$ , but when the base is  
 111 anything other than  $q$ , it will be displayed.

112 We also make use of the  $q$ -binomial coefficients given by

$$113 \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q)_n}{(q)_m (q)_{n-m}}, \quad \text{for } 0 \leq m \leq n \quad (1.5)$$

115 which are polynomials in  $q$  of degree  $m(n-m)$ . When  $n \geq 0$ , the  $q$ -binomial coefficients have value 0 when  
 116  $m < 0$  or when  $m > n$ . This is because  $1/(q)_j = 0$  when  $j < 0$ .

117 We shall often use the following identity involving the  $q$ -binomial coefficients:

$$118 \quad (-cq)_n = \sum_{k \geq 0} c^k q^{T_k} \begin{bmatrix} n \\ k \end{bmatrix}, \quad (1.6)$$

120 where, here and throughout,  $T_k = k(k+1)/2$  is the  $k$ -th Triangular number. In some instances, the expressions  
 121 involving the  $q$ -Pochhammer symbol and the  $q$ -binomial coefficients presented in this paper remain valid with  
 122  $n < 0$ . In such cases, the meaning of  $\begin{bmatrix} n \\ k \end{bmatrix}$  and  $(-cq)_n$  is as [23]. Furthermore, (1.6) remains valid for any integer  
 123  $n$  (even  $n < 0$ ) as can be seen from [23, Thm 4.10].

## 124 2 A very short proof of the key identity for Schur's theorem

125 The proof of the key identity (1.3) given in [9] utilizes the  $q$ -Chu–Vandermonde summation. A second proof  
 126 of (1.3) was given in [9] by rewriting it suitably and then using a Durfee rectangles argument. In October 2022,  
 127 the second author communicated [4] to George Andrews a very short proof of the key identity (1.3), which  
 128 we give here because this proof and the underlying combinatorics motivate the construction of the polynomial  
 129 identity that provides the unification:

130 *Proof* Begin by expanding  $(-aq)_\infty$  and by splitting  $(-bq)_\infty$  as follows:

$$131 \quad (-aq)_\infty(-bq)_\infty = \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} (-bq)_i (-bq^{i+1})_\infty. \quad (2.1)$$

133 Next, expand  $(-bq)_i$  and  $(-bq^{i+1})_\infty$ , and substitute these expansions in (2.1) to get

$$134 \quad (-aq)_\infty(-bq)_\infty = \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} \left( \sum_{j=0}^i b^j q^{T_j} \begin{bmatrix} i \\ j \end{bmatrix} \right) \left( \sum_{\ell=0}^{\infty} \frac{b^\ell q^{T_{\ell+i\ell}}}{(q)_\ell} \right)$$

$$135 \quad = \sum_{i=0}^{\infty} a^i q^{T_i} \left( \sum_{j=0}^i \frac{b^j q^{T_j}}{(q)_j (q)_{i-j}} \right) \left( \sum_{\ell=0}^{\infty} \frac{b^\ell q^{T_{\ell+i\ell}}}{(q)_\ell} \right). \quad (2.2)$$

137 At this stage, consider the following replacements in (2.2)

$$138 \quad j \mapsto \gamma, \quad i - j = i - \gamma \mapsto \alpha, \quad \text{and} \quad \ell \mapsto \beta, \quad (2.3)$$

140 to rewrite the expression on the right in (2.2) as

$$141 \quad \sum_{\alpha, \beta, \gamma} \frac{a^{\alpha+\gamma} b^{\beta+\gamma} q^{T_s+T_\gamma}}{(q)_\alpha (q)_\beta (q)_\gamma}, \quad (2.4)$$

143 where we have used the identity

$$144 \quad T_n + T_m + nm = T_{n+m}$$

146 for Triangular numbers, and  $s = \alpha + \beta + \gamma$ . The key identity follows from (2.1) and (2.4).  $\square$

147 *Remark 2.1* In [9], the first two steps in the combinatorial proof of the generalized Schur theorem were as  
 148 follows: Start with a vector partition  $\langle \pi_a, \pi_b \rangle$  in which  $\pi_a$  is a partition into  $i$  parts in color  $a$ , all distinct,  
 149 and  $\pi_b$  is a partition into  $j$  parts in color  $b$ , all distinct. Then separate the parts of  $\pi_b$  into those that are  $\leq i$   
 150 in size and those that are  $> (i+1)$  in size. There are six steps in that combinatorial proof, but these first two  
 151 steps correspond to

$$152 \quad \frac{a^i q^{T_i}}{(q)_i} (-bq)_i (-bq^{i+1})_\infty,$$

154 and this motivated the starting point of the short proof of the key identity.

## 155 3 A finite version of Lebesgue's identity

156 The  $q$ -binomial coefficients  $\begin{bmatrix} n \\ m \end{bmatrix}$  have the property that

$$157 \quad \lim_{n \rightarrow \infty} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{(q)_m}. \quad (3.1)$$

159 So, a natural way to construct polynomial analogs of  $q$ -hypergeometric identities is to bring in  $q$ -binomial  
 160 coefficients in place of terms like  $1/(q)_m$ . We now establish a polynomial version of Lebesgue's identity<sup>1</sup>:

<sup>1</sup> An equivalent identity appears in [30] in a different form. However, our approach is different.



161 **Theorem 3.1** For all positive integers  $M$ , we have

162 
$$\sum_{i=0}^M q^{T_i} (-cq)_i \begin{bmatrix} M \\ i \end{bmatrix} = (-q)_M \left( \sum_{k=0}^M \frac{c^k q^{2T_k}}{(q^2; q^2)_k} (q^{M-k+1})_k \right).$$

163

164 *Proof* Begin by expanding  $(-cq)_i$  and using (1.5) to rewrite the left-hand side of the expression in Theorem  
165 3.1 as

166 
$$\sum_{i=0}^M q^{T_i} (-cq)_i \begin{bmatrix} M \\ i \end{bmatrix} = \sum_{i=0}^M q^{T_i} \left( \sum_{k=0}^i c^k q^{T_k} \begin{bmatrix} i \\ k \end{bmatrix} \right) \begin{bmatrix} M \\ i \end{bmatrix} = \sum_{i,k} c^k q^{T_i+T_k} \frac{(q)_M}{(q)_k (q)_{i-k} (q)_{M-i}}. \quad (3.2)$$

167

168 If we set  $I = i - k$ , we may rewrite the right-hand side of (3.2) as

169 
$$\sum_{I,k} c^k q^{T_{I+k}+T_k} \frac{(q)_M}{(q)_k (q)_I (q)_{M-I-k}} = \sum_{I,k} c^k q^{T_{I+k}+T_k} \begin{bmatrix} M-I \\ k \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix} \quad (3.3)$$

170

171 again by (1.5). At this stage, we replace  $T_{I+k}$  in (3.3) with  $T_I + T_k + Ik$  to rewrite (3.3) as

172 
$$\sum_{I,k} c^k q^{2T_k+T_I+Ik} \begin{bmatrix} M-I \\ k \end{bmatrix} \begin{bmatrix} M \\ I \end{bmatrix} = \sum_{I,k} c^k q^{2T_k+T_I+Ik} \begin{bmatrix} M \\ k \end{bmatrix} \begin{bmatrix} M-k \\ I \end{bmatrix}, \quad (3.4)$$

173

174 using (1.5) once more. Finally, we write the right-hand side of (3.4) as

175 
$$\sum_k c^k q^{2T_k} \begin{bmatrix} M \\ k \end{bmatrix} \left( \sum_I q^{T_I+Ik} \begin{bmatrix} M-k \\ I \end{bmatrix} \right) = \sum_k c^k q^{2T_k} \begin{bmatrix} M \\ k \end{bmatrix} (-q^{k+1})_{M-k}$$

176 
$$= (-q)_M \sum_{k=0}^M \frac{c^k q^{2T_k}}{(q^2; q^2)_k} \frac{(q)_M}{(q)_{M-k}} = (-q)_M \sum_{k=0}^M \frac{c^k q^{2T_k}}{(q^2; q^2)_k} (q^{M-k+1})_k,$$

177

178 and this proves Theorem 3.1. □

179 *Remark 3.2* Ole Warnaar has pointed out (private correspondence, 2025) that Theorem 3.1 is a special case  
180 of one of Jackson's  $2\phi_1$  transformations, by setting  $c = 0$ ,  $a = q^{-n}$  and replacing  $z$  by  $zq^n$  in Gasper and  
181 Rahman [25, III.4]. Our emphasis has been to provide direct proofs of this and other identities in this paper.

182 *Lebesgue's identity as a limiting case of Theorem 3.1:*

183 Let  $M \rightarrow \infty$  in Theorem 3.1. Then, in view of (3.1), the left-hand side of Theorem 3.1 is

184 
$$\sum_{i=0}^{\infty} \frac{q^{T_i} (-cq)_i}{(q)_i}, \quad (3.6)$$

185

186 which is the left-hand side of (1.1). On the other hand, when  $M \rightarrow \infty$ , the right-hand side of Theorem 3.1  
187 becomes

188 
$$(-q)_{\infty} \sum_{k=0}^{\infty} \frac{c^k q^{2T_k}}{(q^2; q^2)_k} = (-q)_{\infty} (-cq^2; q^2)_{\infty}, \quad (3.7)$$

189

190 because  $(q^{M-k+1})_k \rightarrow 1$  as  $M \rightarrow \infty$ . This yields Lebesgue's identity.

191 *Another finite version of Lebesgue's identity:*

192 There are several possible finite versions of Lebesgue's identity, such as

193 
$$\sum_{N=0}^{m+n} q^{(N^2+N)/2} \sum_{k=0}^N b^k q^{(k^2+k)/2} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ N-k \end{bmatrix} = (-q)_n (q)_m \sum_{i=0}^m \frac{b^i q^{i^2+i} (-q^{n+1})_i}{(q^2; q^2)_i (q)_{m-i}} \quad (3.8)$$

194

which is due to Alladi (1994, unpublished), but we have emphasized the finite version in Theorem 3.1 because this is connected to the unified approach to the Schur and Lebesgue identities that will be given below.

To realize that (3.8) is a finite version of Lebesgue's identity, let  $m, n \rightarrow \infty$  in (3.8). Then the right-hand side (3.8) becomes

$$(-q)_\infty(q)_\infty \sum_{i=0}^{\infty} \frac{b^i q^{i^2+i}}{(q^2; q^2)_i (q)_\infty} = (-q)_\infty(-bq^2; q^2)_\infty, \quad (3.9)$$

which is the right-hand side of (1.1). Under these limits, the left side of (3.8) becomes

$$\sum_{N=0}^{\infty} q^{T_N} \sum_{k=0}^N \frac{b^k q^{T_k}}{(q)_k (q)_{N-k}} = \sum_{N=0}^{\infty} \frac{q^{T_N}}{(q)_N} \sum_{k=0}^N b^k q^{T_k} \begin{bmatrix} N \\ k \end{bmatrix} = \sum_{N=0}^{\infty} \frac{q^{T_N} (-bq)_N}{(q)_N}, \quad (3.10)$$

which is the left-hand side of (1.1).

We now sketch the proof of (3.8) which is non-trivial.

**Proof of (3.8):** In Alladi [3], the following was established both hypergeometrically and combinatorially:

**Lemma 3.3** (Transformation formula)

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2} (-bq; q^2)_n}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{(ab)^n q^{2n^2} (-aq^{2n+1}; q^2)_\infty}{(q^2; q^2)_n}.$$

To prove the Lemma, expand  $(-bq; q^2)_n$  on the left-hand side using (1.6) and reverse the order of summation to get the right-hand side. The combinatorial proof of Lemma 3.3 uses a redistribution idea of Bressoud (see [20, 21]).

The finite version of Lemma 3.3 is

**Lemma 3.4**

$$\sum_{N=0}^{m+n} a^N q^{N^2} \sum_{k=0}^n b^k q^{k^2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ N-k \end{bmatrix}_{q^2} = \sum_{i=0}^m (ab)^i q^{2i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} (-aq^{2i+1}; q^2)_n.$$

To prove Lemma 3.4, expand  $(-aq^{2i+1}; q^2)_n$  to rewrite the right-hand side as

$$\sum_{i=0}^m (ab)^i q^{2i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \sum_{j=0}^n a^j q^{j^2+2ij} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}. \quad (3.11)$$

If we rearrange the right-hand side of (3.11) by putting  $i + j = N$ , we get the left-hand side of Lemma 3.4, thereby proving it.

Next in Lemma 3.4, replace  $a$  by  $aq$  and  $b$  by  $bq$  to get

$$\sum_{N=0}^{m+n} a^N q^{N^2+N} \sum_{k=0}^N b^k q^{k^2+k} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \begin{bmatrix} N \\ N-k \end{bmatrix}_{q^2} = \sum_{i=0}^m (ab)^i q^{2i^2+2i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} (-aq^{2i+2}; q^2)_n. \quad (3.12)$$

In (3.12) replace  $q^2$  with  $q$  to get

$$\sum_{N=0}^{m+n} a^N q^{T_N} \sum_{k=0}^N b^k q^{T_k} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ N-k \end{bmatrix} = \sum_{i=0}^m (ab)^i q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix} (-aq^{i+1})_n. \quad (3.13)$$

If we set  $a = 1$  in (3.13), the right-hand side becomes

$$(q)_m \sum_{i=0}^m \frac{b^i q^{i^2+i} (-q^{i+1})_n}{(q)_i (q)_{m-i}} = (-q)_n (q)_m \sum_{i=0}^m \frac{b^i q^{i^2+i} (-q^{n+1})_i}{(q^2; q^2)_i (q)_{m-i}}, \quad (3.14)$$

and so (3.13) and (3.14) yield (3.8).



## 231 4 A power series identity unifying Schur and Lebesgue

232 Here we prove the following power series identity due to Alamoudi, from which (1.1) and (1.3) emerge as  
233 special cases:

234 **Theorem 4.1** *With free parameters  $a, b, c$ , we have*

$$235 \quad \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} \left(-\frac{c}{a}q\right)_i (-bq^{i+1})_{\infty} = \sum_{t,j,\ell} \frac{a^t b^{\ell} c^j q^{T_{t+\ell+j}+T_j}}{(q)_t(q)_{\ell}(q)_j}.$$

237 *Proof* We expand  $(-\frac{c}{a}q)_i$  and  $(-bq^{i+1})_{\infty}$  to get

$$238 \quad \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} \left(-\frac{c}{a}q\right)_i (-bq^{i+1})_{\infty} = \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} \left( \sum_{j=0}^i \left(\frac{c}{a}\right)^j q^{T_j} \begin{bmatrix} i \\ j \end{bmatrix} \right) \left( \sum_{\ell=0}^{\infty} \frac{b^{\ell} q^{T_{\ell+i}+\ell}}{(q)_{\ell}} \right)$$

$$239 \quad = \sum_{i=0}^{\infty} a^i q^{T_i} \left( \sum_{j=0}^i \frac{(\frac{c}{a})^j q^{T_j}}{(q)_j (q)_{i-j}} \right) \left( \sum_{\ell=0}^{\infty} \frac{b^{\ell} q^{T_{\ell+i}+\ell}}{(q)_{\ell}} \right). \quad (4.1)$$

241 Now, for the sum on the right in (4.1), put  $t = i - j$  and simplify to get

$$242 \quad \sum_{t,j,\ell} \frac{a^t b^{\ell} c^j q^{T_{t+j}+T_j+T_{\ell}+(t+j)\ell}}{(q)_t(q)_j(q)_{\ell}} = \sum_{t,j,\ell} \frac{a^t b^{\ell} c^j q^{T_{t+j+\ell}+T_j}}{(q)_t(q)_j(q)_{\ell}},$$

244 which proves Theorem 4.1.

245 We record two corollaries to Theorem 4.1: □

246 **Corollary 4.2** *The key-identity (1.3) for the generalized Schur's theorem holds.*

247 *Proof* In Theorem 4.1, take  $c = ab$ . Then, the left-hand side of Theorem 4.1 is

$$248 \quad (-bq)_{\infty} \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} = (-aq)_{\infty} (-bq)_{\infty}. \quad (4.2)$$

250 The right-hand side of Theorem 4.1 is

$$251 \quad \sum_{t,j,\ell} \frac{a^{t+j} b^{\ell+j} q^{T_{t+j+\ell}+T_j}}{(q)_t(q)_j(q)_{\ell}}. \quad (4.3)$$

253 Now, (1.3) follows from (4.2) and (4.3) with the replacements

$$254 \quad t \mapsto \alpha, \quad \ell \mapsto \beta, \quad \text{and} \quad j \mapsto \gamma.$$

255 Hence Corollary 4.2. □

257 **Corollary 4.3** *The Lebesgue identity (1.1) holds.*

258 *Proof* Take  $b = 0, a = 1$  in Theorem 4.1. Then, the left-hand side of Theorem 4.1 is the left-hand side of  
259 (1.1). Since  $b = 0$ , the only contribution to the right-hand side of Theorem 4.1 is from  $\ell = 0$ , interpreting  
260  $b^0 = 1$  always. So the right-hand side replacing  $t \mapsto i$  is

$$261 \quad \sum_{i,j} \frac{c^j q^{T_{i+j}+T_j}}{(q)_i(q)_j} = \sum_{i,j} \frac{c^j q^{T_i+2T_j+ij}}{(q)_i(q)_j}$$

$$262 \quad = \sum_{j=0}^{\infty} \frac{c^j q^{2T_j}}{(q)_j} \left( \sum_{i=0}^{\infty} \frac{q^{T_i+ij}}{(q)_j} \right) = \sum_{j=0}^{\infty} \frac{c^j q^{2T_j}}{(q)_j} (-q^{j+1})_{\infty}$$

$$263 \quad = (-q)_{\infty} \sum_{j=0}^{\infty} \frac{c^j q^{2T_j}}{(q^2; q^2)_j} = (-q)_{\infty} (-cq^2; q^2)_{\infty},$$

265 and this yields (1.1). Hence Corollary 4.3. □

266 **Remark 4.4** Since the Schur key-identity and the Lebesgue identity fall out as corollaries (but as two different  
 267 special cases), Theorem 4.1 provides the unification stressed at the beginning of the paper. This unification  
 268 is facilitated by the introduction of a third *free parameter*  $c$  in Theorem 4.1. In [9] where the key-identity is  
 269 proved, the symbol  $c$  is used to denote parts of secondary color, but  $c$  was always taken to be  $ab$  to get the  
 270 product on the right in (1.3). The motivation to introduce the third parameter  $c$  in Theorem 4.1 is from the  
 271 short proof of the key identity; the split product

$$272 \quad 273 \quad (-bq)_i(-bq^{i+1})_\infty$$

274 in the short proof is replaced in Theorem 4.1 by the more general split product

$$275 \quad 276 \quad \left(-\frac{c}{a}q\right)_i(-bq^{i+1})_\infty.$$

277 The above coincide when  $c = ab$ . Now, again, consider the bijection in [9]. The sub-partition  $\lambda_b$  of  $\pi_b$   
 278 containing the parts  $\leq \nu(\pi_a)$  in size is used to construct a new partition  $\pi_{a,c} = \pi_a + (\lambda_b)^*$  with  $\nu(\pi_{a,c}) = \nu(\pi_a)$   
 279 but now  $\nu(\lambda_b)$  of the parts have become  $c$  parts.<sup>2</sup> This motivates the  $\frac{c}{a}$  factor.

280 **Remark 4.5** Ramamani and Venkatachaliengar [29] generalized Lebesgue's identity (1.1) as follows:

$$281 \quad \sum_{i=0}^{\infty} \frac{t^i q^{T_i} (z)_i}{(q)_i} = (z)_\infty (-tq)_\infty \sum_{j=0}^{\infty} \frac{z^j}{(q)_j (-tq)_j}. \quad (4.4)$$

283 Identity (4.4) can be proved  $q$ -hypergeometrically or combinatorially using vector partitions. Now (1.1) can  
 284 be deduced from (4.4) as follows: Take  $t = 1$  and  $z = -cq$ . Then the left-hand side of (4.4) is the left-hand  
 285 side of (1.1). With these values of  $z$  and  $t$ , the right-hand side of (4.4) is

$$286 \quad (-cq)_\infty (-q)_\infty \sum_{j=0}^{\infty} \frac{(-cq)^j}{(q^2; q^2)_j} = \frac{(-cq)_\infty (-q)_\infty}{(-cq; q^2)_\infty} = (-q)_\infty (-cq^2; q^2)_\infty,$$

288 which is the right-hand side of (1.1). Thus Lebesgue's identity follows from (4.4), but is different from our  
 289 derivation of (1.1) from Theorem 4.1, because we get  $(-cq^2; q^2)_\infty$  directly, whereas from (4.4),  $(-cq^2; q^2)_\infty$   
 290 is obtained from the cancellation in

$$291 \quad 292 \quad \frac{(-cq)_\infty}{(-cq; q^2)_\infty}.$$

293 **Remark 4.6** The replacement  $c \mapsto cq^{-1}$  in (1.1) yields the equivalent identity

$$294 \quad \sum_{i=0}^{\infty} \frac{q^{T_i} (-c)_i}{(q)_i} = \prod_{m=1}^{\infty} (1 + q^m)(1 + cq^{2m-1}) = \prod_{m=1}^{\infty} \frac{(1 + cq^{2m-1})}{(1 - q^{2m-1})}. \quad (4.5)$$

296 whose combinatorial interpretation yields Sylvester's famous refinement [31] of Euler's theorem. Ramamani  
 297 and Venkatachaliengar actually generalize (4.5) by establishing an identity equivalent to (4.4); we have pre-  
 298 fered the version of their identity as in (4.4) in view of the discussion of Lebesgue's identity in this paper.

299 **Remark 4.7** Since the right-hand side in Theorem 4.1 is symmetric under interchanging  $a$  and  $b$ , it follows  
 300 that

$$301 \quad \sum_{i=0}^{\infty} \frac{a^i q^{T_i}}{(q)_i} \left(-\frac{c}{a}q\right)_i (-bq^{i+1})_\infty = \sum_{i=0}^{\infty} \frac{b^i q^{T_i}}{(q)_i} \left(-\frac{c}{b}q\right)_i (-aq^{i+1})_\infty.$$

<sup>2</sup>  $\pi^*$  denotes the conjugate of  $\pi$ .



303 **Remark 4.8** We alert the reader that the dilation and translations in (1.4), as well as the translation  $c \rightarrow cq^{-3}$ ,  
 304 the right-hand side of Theorem 4.1 is the sum over generating functions of Schur partitions with the powers  
 305 of  $a$  counting the parts  $\equiv 1$  (mod 3), the powers of  $b$  counting the parts  $\equiv 2$  (mod 3) and the powers of  $c$   
 306 counting the parts divisible by 3. More generally, the function

$$307 \quad 308 \quad G(t, \ell, j) = \frac{q^{T_{t+\ell+j}+T_j}}{(q)_t(q)_\ell(q)_j}$$

309 counts *Type-1* partitions of a prescribed number of parts of each color, specifically,  $t$   $a$ -parts,  $\ell$   $b$ -parts, and  
 310  $j$   $c$ -parts (see [10]). *Type-1* refers to a general class of colored partitions, whose exact definition is given in  
 311 [10], that amount to Schur partitions under standard transformations. Furthermore, in [10], Alladi–Gordon  
 312 demonstrated that there are six schemes (i.e., Type-2 up to Type-6), all counted by  $G(t, \ell, j)$ .

313 In the next section, we shall establish a polynomial version of Theorem 4.1.

## 314 5 A general polynomial identity in three parameters

315 In this section, we prove a general multi-parameter polynomial identity due to Alamoudi, from which some  
 316 of the key results stated above follow either as limiting cases or as special cases.

317 **Theorem 5.1** (Finite three-parameter Schur) *For any pair of integers  $L, M$ , and parameters  $a, b, c$ , we have*

$$318 \quad 319 \quad \sum_{i \geq 0} a^i q^{T_i} \left(-\frac{c}{a}q\right)_i (-bq^{i+1})_{L-i} \begin{bmatrix} M \\ i \end{bmatrix} = \sum_{i, j, k \geq 0} a^i b^j c^k q^{T_{i+j+k}+T_k} \begin{bmatrix} M-i \\ k \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} L-i-k \\ j \end{bmatrix}.$$

320 *Proof* Expand  $\left(-\frac{c}{a}q\right)_i$  and  $(-bq^{i+1})_{L-i}$  to rewrite the left side of Theorem 5.1 as

$$321 \quad \sum_{i \geq 0} a^i q^{T_i} \left(-\frac{c}{a}q\right)_i (-bq^{i+1})_{L-i} \begin{bmatrix} M \\ i \end{bmatrix} \\ 322 \quad = \sum_{i \geq 0} a^i q^{T_i} \left( \sum_{k=0}^i \left(\frac{c}{a}\right) q^{T_k} \begin{bmatrix} i \\ k \end{bmatrix} \right) \left( \sum_{j \geq 0} b^j q^{T_{j+i}} \begin{bmatrix} L-i \\ j \end{bmatrix} \right) \begin{bmatrix} M \\ i \end{bmatrix} \\ 323 \quad = \sum_{\substack{i, j, k \geq 0 \\ k \leq i}} a^{i-k} b^j c^k q^{T_{i+j}+T_k} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix} \\ 324 \quad = \sum_{\substack{i, j, k \geq 0 \\ k \leq i}} a^{i-k} b^j c^k q^{T_{i+j}+T_k} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} M-(i-k) \\ k \end{bmatrix} \begin{bmatrix} M \\ i-k \end{bmatrix}, \quad (5.1)$$

325 because

$$327 \quad 328 \quad \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix} = \begin{bmatrix} M-i+k \\ k \end{bmatrix} \begin{bmatrix} M \\ i-k \end{bmatrix}, \quad (5.2)$$

329 Theorem 5.1 follows by replacing  $i$  by  $i+k$  in (5.1).  $\square$

330 **Remark 5.2** Another way to write the right-hand side of Theorem 5.1 to make it more appealing combinatorially  
 331 and symmetric is to replace

$$332 \quad 333 \quad M \mapsto M' - j - k, \quad (5.3)$$

334 which converts it to

$$335 \quad 336 \quad \sum_{i, j, k} a^i b^j c^k q^{T_{i+j+k}+T_k} \begin{bmatrix} M'-(i+j+k) \\ k \end{bmatrix} \begin{bmatrix} M'-(j+k) \\ i \end{bmatrix} \begin{bmatrix} L-(i+k) \\ j \end{bmatrix}. \quad (5.4)$$

337 We now consider the consequences of Theorem 5.1.

338 **Corollary 5.3** *Theorem 4.1 holds.*

339 *Proof* Let  $M, L \rightarrow \infty$  in Theorem 5.1 to get Theorem 4.1. □

340 **Corollary 5.4** *The following finite (polynomial) version of Lebesgue's identity holds:*

$$341 \quad \sum_{i=0}^{\infty} q^{T_i} (-cq)_i \begin{bmatrix} M \\ i \end{bmatrix} = \sum_{i,k} c^k q^{T_i+2T_k+ik} \begin{bmatrix} M-i \\ k \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix}$$

343 *Proof* Take  $b = 0, a = 1$  in Theorem 5.1. Since  $b = 0$ , the values  $j > 0$  do not make a contribution. Thus, 344 we put  $j = 0$ . This yields Corollary 5.4. □

345 **Remark 5.5** The left-hand side of Corollary 5.4 is identical to the left-hand side of Theorem 3.1. But the 346 right-hand side of Corollary 5.4 is very different from the right-hand side of Theorem 3.1. Thus, Corollary 5.4 347 provides a different finite version of Lebesgue's identity. However, the right-hand side of Corollary 5.4 can be 348 transformed into the right-hand side of Theorem 3.1. In fact, the left side of (3.4) is identical to the right-hand 349 side of Corollary 5.4.

350 **Corollary 5.6** *Theorem 3.1 follows from Theorem 5.1.*

351 To realize that Corollary 5.4 is indeed a finite version of Lebesgue's identity (1.1), let  $M \rightarrow \infty$  in Corollary 352 5.4. Then, the left-hand side of Corollary 5.4 is clearly

$$353 \quad \sum_{i=0}^{\infty} \frac{q^{T_i} (-cq)_i}{(q)_i}$$

355 which is the left-hand side of (1.1). When  $M \rightarrow \infty$ , the right-hand side of Corollary 5.4 becomes

$$356 \quad \sum_{i,k} \frac{c^k q^{T_i+2T_k+ik}}{(q)_i (q)_k} = \sum_{k=1}^{\infty} \frac{c^k q^{2T_k}}{(q)_k} \sum_{i=0}^{\infty} \frac{q^{T_i+ik}}{(q)_i}$$

$$357 \quad = \sum_{k=1}^{\infty} \frac{c^k q^{2T_k}}{(q)_k} (-q^{k+1})_{\infty} = (-q)_{\infty} \sum_{k=1}^{\infty} \frac{c^k q^{2T_k}}{(q^2; q^2)_k} = (-q)_{\infty} (-cq^2; q^2)_{\infty},$$

358 which is the right-hand side of (1.1).

360 **Remark 5.7** We point out that the product of the two  $q$ -binomial coefficients in Corollary 5.4 can be rewritten 361 as

$$362 \quad \begin{bmatrix} M-i \\ k \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix} = \begin{bmatrix} M \\ i, k, M-i-k \end{bmatrix} = \frac{(q)_M}{(q)_i (q)_k (q)_{M-i-k}}, \quad (5.5)$$

364 a  $q$ -multinomial<sup>3</sup> coefficient of order 3. In [10], Alladi–Gordon discuss how the generalized Schur partitions 365 are related to  $q$ -multinomial coefficients of order 3. This link of the Lebesgue identity with  $q$ -multinomial 366 coefficients of order 3 is yet another Schur–Lebesgue unification. In Alladi–Berkovich [7, Eqn. 1.15], a finite 367 version of Lebesgue's identity is established; that identity has the product of two  $q$ -binomial coefficients as in 368 (5.5), but the link between Lebesgue's identity with  $q$ -multinomial coefficients of order 3 and Schur's theorem 369 is not considered in [7]. Warnaar [33] has provided a new proof of the Alladi–Berkovich finite version of the 370 Lebesgue identity.

371 **Remark 5.8** In [6], Alladi–Berkovich prove both combinatorially and  $q$ -theoretically the following double 372 bounded version of the Alladi–Gordon key-identity for Schur's partition theorem:

$$373 \quad q^{T_i+T_j} \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} M-j \\ i \end{bmatrix} = \sum_k q^{T_{i+j-k}+T_k} \begin{bmatrix} M-i-j+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix}. \quad (5.6)$$

<sup>3</sup> In the case of the  $q$ -multinomial coefficient of order 3, such as in (5.5), we display all three indices  $i, k$  and  $M-i-k$  whereas for the  $q$ -binomial coefficient  $\begin{bmatrix} M \\ i \end{bmatrix}$  we suppress  $M-i$  (this is standard notation; see for example [11]).



If we multiply both sides of (5.6) by  $a^i b^j$  and sum over  $i, j$ , we get a double bounded version of (1.3). Our Theorem 5.1 has a third parameter  $c$  and is different in shape from (5.6), but with certain special choices and substitutions, Theorem 5.1 will yield (5.6). In particular, if we set  $c = ab$  in Theorem 5.1 and replace  $i \rightarrow i'$  and  $j \rightarrow j'$ , then we get

$$(-aq)_M(-bq)_L = \sum_{i', j', k} a^{i'} b^{j'} (ab)^k q^{T_{i'+j'+k}+T_k} \begin{bmatrix} M - i' \\ k \end{bmatrix} \begin{bmatrix} M \\ i' \end{bmatrix} \begin{bmatrix} L - i' - k \\ j' \end{bmatrix}. \quad (5.7)$$

In (5.7), by setting  $i = i' + k$ ,  $j = j' + k$ , and comparing the coefficients of  $a^i b^j$  we get

$$\sum_{\substack{i', j', k \\ i'+k=i, j'+k=j}} q^{T_{i'+j'+k}+T_k} \begin{bmatrix} M - i' \\ k \end{bmatrix} \begin{bmatrix} M \\ i' \end{bmatrix} \begin{bmatrix} L - i' - k \\ j' \end{bmatrix} = q^{T_i+T_j} \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} M \\ i \end{bmatrix}. \quad (5.8)$$

Setting  $M \rightarrow M - j$  in (5.8) and simplifying the left-hand side yields (5.6). Notice, however, that the double bounded version of (1.3) obtained by multiplying both sides of (5.6) by  $a^i b^j$  and summing over  $i, j$  is different from (5.7).

*Remark 5.9* For  $M \geq 0$ , setting  $b = 0$  in Theorem 5.1 gives

$$\sum_{i=0}^M a^i q^{T_i} \left(-\frac{c}{a} q\right)_i \begin{bmatrix} M \\ i \end{bmatrix} = \sum_{i, k} a^i c^k q^{T_{i+k}+T_k} \begin{bmatrix} M \\ i, k, M - i - k \end{bmatrix}.$$

Letting  $q \rightarrow 1$  gives

$$\sum_{i=0}^M a^i \left(1 + \frac{c}{a}\right)^i \binom{M}{i} = \sum_{i, k} a^i c^k \binom{M}{i, k, M - i - k}.$$

The left-hand side is

$$\sum_{i=0}^M (a + c)^i \binom{M}{i} = (1 + a + c)^M.$$

The above is an instance of the trinomial theorem.

## 6 A key identity for the Alladi–Schur Theorem

Schur's partition theorem has always been associated with the modulus 3 or the modulus 6, the latter because

$$\prod_{m=1}^{\infty} (1 + q^{3m-2})(1 + q^{3m-1}) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^{6m-5})(1 - q^{6m-1})}. \quad (6.1)$$

Sometime during the 90s, the second author noted that

$$\prod_{m=1}^{\infty} \frac{1}{(1 - q^{6m-5})(1 - q^{6m-1})} = \prod_{m=1}^{\infty} (1 + q^{2m-1} + q^{4m-2}), \quad (6.2)$$

where the second product in (6.2) is the generating function of  $A(n)$ , the number partitions of  $n$  into odd parts repeating no more than twice, and suggested to George Andrews that it would be worthwhile to explore the deeper connections between the equality

$$S(n) = A(n). \quad (6.3)$$

Andrews dubbed the equality in (6.3) the *Alladi–Schur Theorem* and established in [14] the following deep refinement:

411 **Theorem A** (Andrews' refinement of the Alladi–Schur theorem)

412 Let  $A(n; k)$  denote the number of partitions of  $n$  into odd parts repeating no more than twice and with exactly  
413  $k$  parts.

414 Let  $s(n; k)$  denote the number of partitions of  $n$  into parts that differ by  $\geq 3$ , and with no consecutive  
415 multiples of 3, and having exactly  $k$  parts, where the even parts are counted twice. Then

416 
$$A(n; k) = s(n; k). \quad (6.4)$$

418 **Remark 6.1** What is surprising is that in Theorem A, partitions of the Schur type are classified according to  
419 their parity. Thus, the equality (6.2) has provided a fresh direction for the investigation of Schur partitions.  
420 Andrews' proof of Theorem A [14] was  $q$ -theoretic. In view of the combinatorial elegance of Theorem A, it is  
421 natural to ask if there is a combinatorial/bijective proof; such a proof was recently found by Alamoudi in [1],  
422 and it turned out to be quite intricate. Some notions in [1] share a resemblance with, but are different from,  
423 some of the notions in Kurşungöz's important paper [28] on Schur's partition theorem.

424 With the combinatorial proof of Theorem A having been found, the following question arises. Can Theorem  
425 A be cast in the form of a  $q$ -hypergeometric key identity? We answer this in the affirmative below.

426 In [28], Kurşungöz obtains, by combinatorial arguments, a series generating function for the Schur partitions,  
427 which is different from the series in the Alladi–Gordon key identity; then, by the same combinatorial  
428 arguments, he obtains a series generating function for Schur partitions by keeping track of the number of even  
429 and odd parts. His result is:

430 **Theorem K** Let  $s(n; m_1, m_0)$  denote the number of Schur partitions of  $n$  having  $m_1$  odd parts and  $m_0$  even  
431 parts. Then

432 
$$\sum_{m_1, m_0, n \geq 0} s(n; m_1, m_0) a^{m_1} b^{m_0} q^n = \sum_{n_{11}, n_{10}, n_{21}, n_{22} \geq 0} \frac{q^{6n_{21}^2 - n_{21} + 6n_{22}^2 + n_{22} + 2n_{11}^2 - n_{11} + 2n_{10}^2}}{(q^2; q^2)_{n_{11}} (q^2; q^2)_{n_{10}} (q^6; q^6)_{n_{21}} (q^6; q^6)_{n_{22}}} \\ \times q^{12n_{21}n_{22} + 6(n_{21} + n_{22})(n_{11} + n_{10}) + 4n_{11}n_{10}} a^{n_{21} + n_{22} + n_{11}} b^{n_{21} + n_{22} + n_{10}}. \quad (6.5)$$

434 In deriving Theorem K, Kurşungöz groups the Schur partitions of  $n$  into disjoint *pairs*, which are parts that  
435 differ by exactly 3 (with a certain convention when there is a maximal chain of  $\ell$  parts differing by 3 with  $\ell$   
436 is odd), and calls the rest *singletons*. In the above identity,  $n_{11}$  (resp.  $n_{10}$ ) is the number of odd (resp. even)  
437 singletons, and  $n_{21}$  (resp.  $n_{22}$ ) is the number of 1 (mod 3) (resp. 2 (mod 3)) pairs. We now point out that in  
438 view of Andrews' refinement of the Alladi–Schur Theorem and Kurşungöz's series representation (6.5) for the  
439 generating function of  $s(n; m_1, m_0)$ , if we choose

442 
$$b = a^2, \quad (6.6)$$

444 then the series in (6.5) will be equal to the product

445 
$$\prod_{m=1}^{\infty} (1 + aq^{2m-1} + a^2 q^{4m-2}), \quad (6.7)$$

447 and this yields the key identity for Theorem A. That is

448 
$$\sum_{m_1, m_0, n \geq 0} s(n; m_1, m_0) a^{m_1 + 2m_0} q^n = \sum_{n_{11}, n_{10}, n_{21}, n_{22} \geq 0} \frac{q^{6n_{21}^2 - n_{21} + 6n_{22}^2 + n_{22} + 2n_{11}^2 - n_{11} + 2n_{10}^2}}{(q^2; q^2)_{n_{11}} (q^2; q^2)_{n_{10}} (q^6; q^6)_{n_{21}} (q^6; q^6)_{n_{22}}} \\ \times q^{12n_{21}n_{22} + 6(n_{21} + n_{22})(n_{11} + n_{10}) + 4n_{11}n_{10}} a^{3n_{21} + 3n_{22} + n_{11} + 2n_{10}} = \prod_{m=1}^{\infty} (1 + aq^{2m-1} + a^2 q^{4m-2}) \quad (6.8)$$

450 is the *key identity* for Theorem A. This seems to have escaped attention. It is desirable to have a  $q$ -hypergeometric  
451 proof of (6.8). Kurşungöz's method has been used by other authors to obtain new series generating functions for  
452 various fundamental partition functions. But Theorem K had not been considered in conjunction with Theorem  
453 A, which is the reason that the key identity (6.8) for Theorem A presented here had escaped attention. In [13],



456 Andrews has expressed the view that the new direction for Schur's theorem presented by Theorem A is deeper  
 457 and more significant than the classical version of Schur's theorem. This is confirmed by the complexity of  
 458 the key identity (6.8), for which, at the time of this writing, a  $q$ -hypergeometric proof is not known. There is,  
 459 however, another series representation for the generating function of  $s(n; m_1, m_2)$ , due to Andrews–Chern–Li  
 460 [15], which is

$$461 \quad \sum_{n, m_1, m_2} s(n; m_1, m_2) a^{m_1+m_2} b^{m_2} q^n = \sum_{n_1, n_2, n_3 \geq 0} \frac{(-1)^{n_3} a^{n_1+n_2+2n_3} b^{n_2+n_3}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} (q^6; q^6)_{n_3}} \\ 462 \quad \times q^{2n_1^2 - n_1 + 2n_2^2 + 9n_3^2 + 2n_1n_2 + 6n_1n_3 + 6n_2n_3}. \quad (6.9)$$

464 When one sets  $a = b$  in (6.9), which means that the even parts are counted twice, then one can set the resulting  
 465 expression equal to the product on the right in (6.8). Thus, we have the identity

$$466 \quad \sum_{n_1, n_2, n_3 \geq 0} \frac{(-1)^{n_3} a^{n_1+2n_2+3n_3}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} (q^6; q^6)_{n_3}} \times q^{2n_1^2 - n_1 + 2n_2^2 + 9n_3^2 + 2n_1n_2 + 6n_1n_3 + 6n_2n_3} \\ 467 \quad = \prod_{m=1}^{\infty} (1 + aq^{2m-1} + a^2 q^{4m-2}). \quad (6.10)$$

469 Andrews–Chern–Li provide two proofs of (6.10), one  $q$ -hypergeometric, and another which is computer-  
 470 aided. But it is to be noted that in (6.9) and (6.10), there is the factor  $(-1)^{n_3}$ , and so it is not transparent that  
 471 the coefficients in the power series expansion are non-negative. On the other hand, it is transparent that the  
 472 coefficients of the series on the left in (6.8) are all non-negative.

## 473 7 The Capparelli theorems and the key-identity

474 Through a study of vertex operators in Lie algebras, Capparelli [22] conjectured the following partition theorem:

475 **Theorem C** Let  $C^*(n)$  denote the number of partitions of  $n$  into parts  $\equiv \pm 2, \pm 3 \pmod{12}$ .

476 Let  $D(n)$  denote the number of partitions of  $n$  into parts  $> 1$  with minimal difference  $\geq 2$ , where the difference  
 477 is  $\geq 4$  unless consecutive parts are multiples of 3 or add up to a multiple of 6. Then

$$478 \quad C^*(n) = D(n).$$

480 The first proof of Theorem C was due to Andrews [12] by the use of generating functions. Subsequently,  
 481 Alladi–Andrews–Gordon [5] noticed that if  $C^*(n)$  is replaced by the equivalent partition function  $C(n)$ , which  
 482 is the number of partitions of  $n$  into distinct parts  $\equiv 2, 3, 4$  or  $6 \pmod{6}$ , then there is a three-parameter  
 483 refinement, namely:

484 **Theorem C-R** Let  $C(n; i, j, k)$  denote the number of partitions of the type enumerated by  $C(n)$ , with the  
 485 added restriction that there are precisely  $i$  parts  $\equiv 4 \pmod{6}$ ,  $j$  parts  $\equiv 2 \pmod{6}$ , and of those  $\equiv 0$   
 486 ( $\pmod{3}$ ), exactly  $k$  are  $> 3(i + j)$ .

487 Let  $D(n; i, j, k)$  denote the number of partitions of the type enumerated by  $D(n)$  with the additional  
 488 restriction that there are precisely  $i$  parts  $\equiv 1 \pmod{3}$ ,  $j$  parts  $\equiv 2 \pmod{3}$ , and  $k$  parts  $\equiv 0 \pmod{3}$ . Then

$$488 \quad C(n; i, j, k) = D(n; i, j, k).$$

491 Alladi–Andrews–Gordon [5] established a generalization of Theorem C-R by the *method of weighted words*  
 492 (which was initiated in Alladi–Gordon [9] to establish a generalization of Schur's theorem), and viewed this  
 493 generalized theorem as the combinatorial interpretation of the following *key identity*:

$$494 \quad \sum_{i, j} \frac{a^i b^j q^{2T_i+2T_j} (-q)_{i+j} (-cq^{i+j+1})_{\infty}}{(q^2; q^2)_i (q^2; q^2)_j} \\ 495 \quad = \sum_{i, j, k} \frac{a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}} \begin{bmatrix} i+j+k \\ k \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}. \quad (7.1)$$

497 The sum on the right in (7.1) could be rewritten as

$$\begin{aligned}
 498 & \sum_{i,j,k} \frac{a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}} \frac{(q)_{i+j+k}}{(q)_{i+j}(q)_k} \frac{(q^2; q^2)_{i+j}}{(q^2; q^2)_i(q^2; q^2)_j} \\
 499 & = \sum_{i,j} \frac{a^i b^j q^{2T_i+2T_j} (-q)_{i+j}}{(q^2; q^2)_i(q^2; q^2)_j} \sum_k \frac{c^k q^{T_k+(i+j)k}}{(q)_k} \\
 500 & = \sum_{i,j} \frac{a^i b^j q^{2T_i+2T_j} (-q)_{i+j} (-cq^{i+j+1})_\infty}{(q^2; q^2)_i(q^2; q^2)_j}. \tag{7.2}
 \end{aligned}$$

502 If we take  $c = 1$ , then the term in (7.2) becomes

$$\begin{aligned}
 503 & (-q)_\infty \sum_{i,j} \frac{a^i b^j q^{2T_i+2T_j}}{(q^2; q^2)_i(q^2; q^2)_j} = (-aq^2; q^2)_\infty (-bq^2; q^2)_\infty (-q)_\infty. \tag{7.3}
 \end{aligned}$$

505 If we make the replacements

$$\begin{aligned}
 506 & q \mapsto q^3, \quad a \mapsto q^{-2}, \quad b \mapsto q^{-4} \tag{7.4}
 \end{aligned}$$

508 in (7.4), we get the generating function of  $C(n)$  in Theorem C-R.

509 Capparelli [22] had stated another partition conjecture in the form  $A^*(n) = B(n)$ . The difference between  
510 the conditions defining  $B(n)$  and  $D(n)$  is that among the partitions enumerated by  $B(n)$ , the integer 2 should not  
511 occur as a part, but 1 is allowed as part. The generating function of  $A^*(n)$  is a product that is more complicated  
512 than the product generating function of  $C^*(n)$ . However, it turns out that  $A^*(n) = A(n)$ , where  $A(n)$  is the  
513 number of partitions of  $n$  into distinct parts  $\equiv 1, 3, 5, \text{ or } 6 \pmod{6}$ . So, this second conjecture is equivalent  
514 to  $A(n) = B(n)$ . This conjecture can be proved by applying the transformations

$$\begin{aligned}
 515 & q \mapsto q^3, \quad a \mapsto q^{-5}, \quad b \mapsto q^{-1}, \tag{7.5}
 \end{aligned}$$

517 and by combinatorially interpreting the resulting  $q$ -hypergeometric identity. So what we stress here is that in  
518 [5], by considering partitions into *distinct* parts in certain residue classes modulo 6, instead of partitions into  
519 parts in certain residue classes modulo 12 (parts that could repeat), not only is the second partition theorem  
520 of Capparelli cast in a more elegant form, but also that such a reformulation is capable of refinement and  
521 generalization. The idea that reformulating partitions into certain distinct parts is capable of refinements was  
522 initiated in Alladi–Gordon’s treatment of Schur’s theorem [9] and indeed that was instrumental in developing  
523 the *method of weighted words* which is widely applicable.

524 It is to be noted that when we set  $a = 0$ , the generalized Capparelli product on the right-hand side of (7.3)  
525 reduces to

$$\begin{aligned}
 526 & (-bq; q^2)_\infty (-q)_\infty, \tag{7.6}
 \end{aligned}$$

528 which is the product for Lebesgue’s identity. In view of this link between the generalized Capparelli identity  
529 and Lebesgue’s identity, it is natural to ask whether a finite version of the Capparelli key identity can be  
530 obtained from the Transformation Formula (Lemma 3.3) in Sect. 3 just as we obtained the finite version of  
531 Lebesgue’s identity from Lemma 3.3. The answer is YES. Indeed, Alladi (1994, unpublished), obtained the  
532 following from Lemma 3.3: *For positive integers  $m, n$ , we have*

$$\begin{aligned}
 533 & \sum_{j=0}^m \sum_{i=0}^n (bc)^i a^j q^{2T_i+2T_j} \begin{bmatrix} m \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} (-cq^{i+1})_{\ell+j} \\
 534 & = \sum_{N=0}^{m+n+\ell} c^N q^{TN} \left( \sum_{j=0}^n \sum_{i=0}^m b^i a^j q^{T_i+2T_j} \begin{bmatrix} m \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \begin{bmatrix} \ell+j \\ N-i \end{bmatrix}_q \right). \tag{7.7}
 \end{aligned}$$



536 If we let  $m, n, \ell \rightarrow \infty$  in (7.7), then we get

$$537 \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(bc)^i i a^j q^{2T_i+2T_j} (-q)_i (-cq^{i+1})_{\infty}}{(q^2; q^2)_i (q^2; q^2)_j} = (-aq^2; q^2)_{\infty} \sum_{N=0}^{\infty} \frac{c^N q^{T_N} (-bq)_N}{(q)_N}. \quad (7.8)$$

538

539 Identities (7.7) and (7.8) are different from the Capparelli key identity (7.1) in the sense that in (7.1) we have  
540  $(-q)_{i+j} (-cq^{i+j+1})_{\infty}$ , whereas in (7.8) we have  $(-q)_i (-cq^{i+1})_{\infty}$ . However, when  $c = 1$  both versions are  
541 the same, and the expression on (7.8) becomes the product

542

$$543 \quad (-aq^2; q^2)_{\infty} (-bq^2; q^2)_{\infty} (-q)_{\infty}. \quad (7.9)$$

545 But there is a general version of (7.1), which specializes into a polynomial identity, with a free parameter  $c$   
546 where the decomposition involves  $(-q)_{i+j}$  as noted by Alamoudi: *For integers  $M_1, M_2, L$ , we have*

$$547 \quad \sum_{j, i \geq 0} a^i b^j q^{2T_i+2T_j} \begin{bmatrix} M_1 \\ i \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ j \end{bmatrix}_{q^2} (-q)_{i+j} (-cq^{i+j+1})_{L-i-j}$$

$$548 \quad = \sum_{i, j, k \geq 0} a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k} \frac{(q^{L-i-j-k+1})_k (q^{2M_1-2i+2}; q^2)_i (q^{M_2-2j+2}; q^2)_j}{(q)_{i+j+k}} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}$$

549

550 In particular, for non-negative integers  $M_1, M_2, L$ , with  $L \geq M_1 + M_2$  we have

$$551 \quad \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} a^i b^j q^{2T_i+2T_j} \begin{bmatrix} M_1 \\ i \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ j \end{bmatrix}_{q^2} (-q)_{i+j} (-cq^{i+j+1})_{L-i-j}$$

$$552 \quad = \sum_{\substack{i, j, k \geq 0 \\ i \leq M_1, j \leq M_2, i+j+k \leq L}} a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} L \\ i+j+k \end{bmatrix} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}$$

$$553 \quad \times \frac{(q^{2M_1-2i+2}; q^2)_i (q^{M_2-2j+2}; q^2)_j}{(q^{L-i-j+1})_{i+j}}.$$

554

555 After performing various cancellations, the right-hand sides of (7.10) and (7.11) become

$$556 \quad \sum_{i, j, k \geq 0} a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} M_1 \\ i \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} L-i-j \\ k \end{bmatrix} (-q)_{i+j}. \quad (7.12)$$

557

558 If we sum the inner sum over  $k$  and use

$$559 \quad \sum_{k \geq 0} c^k q^{T_k+(i+j)k} \begin{bmatrix} L-i-j \\ k \end{bmatrix} = (-cq^{i+j+1})_{L-i-j}, \quad (7.13)$$

560

561 we get the left-hand sides of (7.10) and (7.11). If we let  $M_1, M_2, L \rightarrow \infty$ , we get the (infinite version)  
562 Capparelli key identity given by (7.1) and (7.2).

563 *Remark 7.1* We have given the intermediate identities (7.10) and (7.11) because they maintain the form of  
564 (7.1), which is the key identity of the original Alladi–Andrews–Gordon three-parameter refinement of the  
565 Capparelli partition theorem. The combinatorial significance of the form in (7.1) is that it highlights the  
566 generating function [5, Eq. (5.5)] which counts the relevant minimal partitions.

567 **Remark 7.2** Given the generality of (7.10), we can apply the same bound shifting technique used to obtain  
 568 (5.6) from Theorem 5.1 and get other finite analogs of (7.1). Specifically, by setting the coefficients of  $a^i b^j$   
 569 equal, we get

$$570 \quad a^i b^j q^{2T_i+2T_j} \begin{bmatrix} M_1 \\ i \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ j \end{bmatrix}_{q^2} (-q)_{i+j} (-cq^{i+j+1})_{L-i-j} \\ 571 \quad = \sum_{k \geq 0} a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k} \frac{(q^{L-i-j-k+1})_k (q^{2M_1-2i+2}; q^2)_i (q^{M_2-2j+2}; q^2)_j}{(q)_{i+j+k}} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2}. \\ 572$$

573 The above equation is valid for any choice of integers  $M_1, M_2, L, i, j$  with  $i, j \geq 0$ . Setting  $M_1 = L - j$ ,  
 574  $M_2 = L$ , canceling using  $(q^{a-j-i+1})_{i+j} = (q^{a-j-i+1})_i (q^{a-j+1})_j$  for any  $a \in \mathbb{Z}$ , and summing for all  
 575  $i, j \geq 0$  gives, for any  $L \in \mathbb{Z}$ ,

$$576 \quad \sum_{i,j \geq 0} a^i b^j q^{2T_i+2T_j} \begin{bmatrix} L-j \\ i \end{bmatrix}_{q^2} \begin{bmatrix} L \\ j \end{bmatrix}_{q^2} (-q)_{i+j} (-cq^{i+j+1})_{L-i-j} \\ 577 \quad = \sum_{i,j,k \geq 0} a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k} \begin{bmatrix} L \\ i+j+k \end{bmatrix} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} (-q^{L-i-j+1})_{i+j}. \\ 578$$

579 The point is in (7.14), the integers  $M_1, M_2, i, j$ , and  $L$  are fixed and only  $k$  is being summed over. Thus, many  
 580 substitutions can be made, perhaps sending  $M_1$  to something that depends on  $j$ , and the expression remains  
 581 valid, after which one can sum over  $i, j \geq 0$ . This way, one may obtain a myriad of other finite analogs of  
 582 (7.1). However, *the same liberty does not necessarily extend to shifting by expressions involving  $k$*  since  $k$  is  
 583 still bound by the summation in (7.14) and is not free as  $i \geq 0$  and  $j \geq 0$  are in (7.14).

584 In closing this section, we note that when we set  $a = 0$  in (7.9), the Capparelli product generating function  
 585 reduces to the product generating function for Lebesgue's identity, and if we further set  $b = 0$ , we get the  
 586 product generating function for Euler's theorem on partitions into distinct parts. This leads us to an infinite  
 587 hierarchy of identities observed by Alladi [3, unpublished], of which the first three cases are those of Euler,  
 588 Lebesgue, and Capparelli, in that order. We present this in the next section, along with a new polynomial  
 589 version of this infinite hierarchy due to Alamoudi.

## 590 8 An infinite hierarchy of $q$ -hypergeometric identities

591 In 1994, Alladi observed that for each non-negative integer  $r$ , there is the identity

$$592 \quad \sum_{v_1, v_2, \dots, v_r, k \geq 0} \frac{a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{2T_{v_1}+2T_{v_2}+\cdots+2T_{v_r}+T_k+k(v_1+v_2+\cdots+v_k)} (-q)_{v_1+v_2+\cdots+v_k}}{(q^2; q^2)_{v_1} (q^2; q^2)_{v_2} \cdots (q^2; q^2)_{v_r} (q)_k} \\ 593 \quad = \sum_{v_1, v_2, \dots, v_r \geq 0} \frac{a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2T_{v_1}+2T_{v_2}+\cdots+2T_{v_r}} (-q)_{v_1+v_2+\cdots+v_r} (-cq^{v_1+v_2+\cdots+v_r+1})_\infty}{(q^2; q^2)_{v_1} (q^2; q^2)_{v_2} \cdots (q^2; q^2)_{v_r}}. \\ 594$$

595 Identity (8.1) is easily proved by summing over  $k$  the inner sum on the left-hand side. When  $c = 1$ , identity  
 596 (8.1) becomes

$$597 \quad \sum_{v_1, v_2, \dots, v_r, k \geq 0} \frac{a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2T_{v_1}+2T_{v_2}+\cdots+2T_{v_r}+T_k+k(v_1+v_2+\cdots+v_k)} (-q)_{v_1+v_2+\cdots+v_k}}{(q^2; q^2)_{v_1} (q^2; q^2)_{v_2} \cdots (q^2; q^2)_{v_r} (q)_k} \\ 598 \quad = (-a_1 q^2; q^2)_\infty (-a_2 q^2; q^2)_\infty \cdots (a_r q^2; q^2)_\infty (-q)_\infty. \\ 599$$

600 Note that the case  $r = 0$  in (8.2) is Euler's identity for partitions into distinct parts, the case  $r = 1$  gives the  
 601 product for Lebesgue's identity, and the case  $r = 2$  yields the product for the generalized Capparelli identity  
 602 (7.3). This was the motivation to come up with this infinite hierarchy:

$$603 \quad \text{Euler } (r = 0) \rightarrow \text{Lebesgue } (r = 1) \rightarrow \text{Capparelli } (r = 2) \rightarrow \cdots. \quad (8.3)$$



605 To get regular partition theorems (not weighted ones) from (8.1) and (8.2), the minimal dilation is

$$606 \quad q \mapsto q^{r+1}, \quad (8.4)$$

608 and this is the optimal *dilation*. There are several choices of residue classes modulo  $r+1 = m$  for this dilation:  
609 Let  $j_1, j_3, \dots, j_r$  be incongruent modulo  $m$ , with  $0 < j_i < 2m$ ,  $j_i \neq m$ . Consider the *translations*

$$610 \quad a_i \mapsto a_i q^{j_i-m} \quad \text{for } i = 1, 2, \dots, r. \quad (8.5)$$

612 We will now turn our attention to the corresponding partition theorem. However, we must first discuss the  
613 important notion of level parities. We begin by alerting the reader to the following convention we adopt in the  
614 sequel.

615 In the sequel, we adopt the following convention. Whenever we say that a part is  $\equiv j \pmod{M}$ , we mean  
616 a part of the form  $j + \lambda M$ , with the integer  $\lambda \geq 0$ . In particular, the part is  $\geq j$ . For example, if a part is  $\equiv 4$   
617  $(\pmod{3})$ , that part must be one of 4, 7, 10, ....

618 For each part  $p \equiv j \pmod{M}$ , such that  $M \nmid j$ , we say that  $p$  has *odd level parity* as a  $j \pmod{M}$  part  
619 if  $p = j + \lambda M$  with  $\lambda$  odd. Otherwise, we say  $p$  has *even level parity* as a  $j \pmod{M}$  part.

620 **Remark 8.1** In the sequel, we will simply say that a part  $p$  has odd/even level parity and omit the prepositional  
621 phrase “as a  $j \pmod{M}$  part” as it will be clear from context. Nonetheless, we would like to alert the reader  
622 to the following subtlety. In general, the level parity of a part  $p$  depends on the specific choice of  $j$  and  $M$ . For  
623 example, 7 has odd level parity if  $j = 2$  and  $M = 5$ , but even level parity if  $j = 1$  and  $M = 3$ . Moreover, in  
624 view of the above convention, even if  $j \equiv j' \pmod{M}$  in the traditional sense,  $j$  and  $j'$  can define different  
625 level parities for  $p$ . For example, 7 has odd level parity if  $j = 4$  and  $M = 3$  but even level parity if  $j = 1$  and  
626  $M = 3$ , even though  $4 \equiv 1 \pmod{3}$ .

627 **Theorem H** *Let  $A(n; v_1, v_2, \dots, v_r; 2m)$  denote the number of partitions of  $n$  into  $v_i$  distinct parts  $\equiv j_i$   
628  $(\pmod{2m})$  and distinct parts  $\equiv 0 \pmod{m}$ , for  $i = 1, 2, \dots, r$ .*

629 *Let  $B(n; v_1, v_2, \dots, v_r; m)$  denote the number of partitions of  $n$  into  $v_i$  distinct parts  $\equiv j_i \pmod{m}$  for  
630  $i = 1, 2, \dots, r$  such that the difference between two parts of different level parities is  $> m$ , and distinct parts  
631  $\equiv 0 \pmod{m}$  each  $> (j_1 + j_2 + \dots + j_r)m$ .*

632 *Then, we have*

$$633 \quad A(n; v_1, v_2, \dots, v_r; 2m) = B(n; v_1, v_2, \dots, v_r; m). \quad (8.6)$$

635 **Remark 8.2** In the combinatorial proofs of Schur’s theorem in Alladi–Gordon [9], and of Capparelli’s theorem  
636 in Alladi–Andrews–Gordon [5], a combinatorial method due to Bressoud is followed. This method involves  
637 two stages: Stage 1 is an *embedding* (of the Ferrers conjugate of the distinct parts  $\equiv 0 \pmod{m}$  which are  
638  $\leq (v_1 + v_2 + \dots + v_r)m$  into the Ferrers graph of the partitions parts into distinct parts  $\equiv j_i \pmod{2m}$ )  
639 for  $i = 1, 2, \dots, r$ . This is followed by Stage 2, which is a rearrangement, which we call the *Bressoud*  
640 *rearrangement*, and this is more complicated. It is only after the rearrangement that we get a partition satisfying  
641 certain difference conditions. The partition function  $A$  in (8.6) is at the product level, and this is the start of  
642 the combinatorial construction. The partition function  $B$  is at the embedding stage of the combinatorial proof.  
643 What we require is the partition function  $C$  (whose parts satisfy difference conditions based on the residue  
644 classes), which results AFTER the Bressoud rearrangement is completed. We expect to present the function  $C$   
645 after working out the details in a subsequent paper [2] devoted to the combinatorics of various identities that  
646 have been presented here. However, we will present  $C$  in a certain special case (see Theorem C5).

647 **Remark 8.3** We note the following with regards to Theorem H.

- 648 (i) It is to be noted that for the partition function  $A$  in Theorem H, the modulus is  $2m$ , whereas for the  
649 partition function  $B$ , the modulus is  $m$ . This halving of the modulus for  $B$  is due to the embedding.
- 650 (ii) Due to observed combinatorial insights and intricacies, the authors have refined the statement of Theorem  
651 H, as well as Theorem C5 presented later in this section, in this version of the manuscript. This includes  
652 the introduction of the notion of level parities. Regarding this notion, it is important to note that although  
653 the difference between two parts of different level parities is  $> m$ , in view of the definition of level parity,  
654 consecutive multiples of  $m$  are allowed for Theorem H.

655 In the case  $r = 1$ , the dilation gives the Little Göllnitz theorems, not the Lebesgue identity which corre-  
656 sponds to the undilated case.

657 For  $r = 1$ , the function  $C$  is as in Corollary 2<sup>A</sup> of Alladi–Gordon [8]. There, the function  $B$  is not mentioned.  
658 It would be  $B(n; j, 2) =$  the number of partitions into  $j$  distinct and non-consecutive odd parts, and even  
659 parts, each  $> 2j$ .

660 In the case  $r = 2$ , which corresponds to Capparelli’s theorem,  $C$  and  $A$  are known, but  $B$  is not mentioned.  
661 It will be  $B(n; v_1, v_2; 3) =$  the number of partitions of  $n$  into distinct  $v_1$  parts  $\equiv 4 \pmod{3}$  and distinct  $v_2$   
662 parts  $\equiv 2 \pmod{3}$  such that the gap between parts of different parities is  $> 3$ , and distinct parts  $\equiv 0 \pmod{3}$   
663 each  $> 3(v_1 + v_2)$ .

664 Actually, there are two theorems of Capparelli (which correspond to the case  $r = 2$ ). For the first theorem,  
665 the function  $A$  has generating function

$$666 \quad (-a_1 q^2; q^6)_\infty (-a_2 q^4; q^6)_\infty (-q^3; q^3)_\infty. \quad (8.7)$$

668 The second theorem of Capparelli corresponds to the generating function of  $A(n)$  being

$$669 \quad (-a_1 q; q^6)_\infty (-a_2 q^5; q^6)_\infty (-q^3; q^3)_\infty. \quad (8.8)$$

671 When we set  $a_1 = a_2 = 1$ , the product in (8.7) becomes

$$672 \quad \frac{1}{(q^3; q^6)_\infty (q^2; q^{12})_\infty (q^{10}; q^{12})_\infty} = \frac{1}{(q^3; q^{12})_\infty (q^9; q^{12})_\infty (q^2; q^{12})_\infty (q^{10}; q^{12})_\infty}. \quad (8.9)$$

674 It is in the unrefined form as in (8.9) that Capparelli stated his conjecture. It was in Alladi–Andrews–Gordon  
675 [5] that the product in (8.9) was replaced by (8.7), which had the advantage that the theorem could be refined  
676 by introducing parameters  $a_1, a_2$ . When we set  $a_1 = a_2 = 1$  in (8.8), it does not yield a product as nice as in  
677 (8.9), but more complicated, and it is such a product that Capparelli stated. Again, the advantage of considering  
678 distinct parts, as pointed out in [5], is that Capparelli’s second theorem can be more neatly expressed in terms of  
679 the product (8.8). So the question that arises is whether, for  $r \geq 3$ , there are such special choices of the residue  
680 classes modulo  $2m = 2(r + 1)$ , where the product in terms of distinct parts determined modulo  $m$  transforms  
681 neatly into a product where the modulus is  $2m$ , and the parts can repeat as in (8.9). Such a phenomenon does  
682 not occur when  $r = 3$ , but does occur for all even  $r \geq 4$ . We illustrate this with an example of  $r = 4$  (so  
683  $m = r + 1 = 5$ ). Thus, as per Theorem L, we need to choose four residue classes  $\pmod{10}$ , which are  
684 incongruent  $\pmod{5}$  and unequal to 0 or 5  $\pmod{10}$ . The ideal choice is 2, 4, 6, 8  $\pmod{10}$ . This then gives

$$685 \quad (-q^2; q^{10})_\infty (-q^4; q^{10})_\infty (-q^6; q^{10})_\infty (-q^8; q^{10})_\infty (-q^5; q^5)_\infty \\ 686 \quad = \frac{1}{(q^5; q^{10})_\infty (q^2; q^{20})_\infty (q^6; q^{20})_\infty (q^{14}; q^{20})_\infty (q^{18}; q^{20})_\infty}, \quad (8.10)$$

688 which is an ideal extension of Capparelli’s first theorem to the level  $r = 4$ . Thus, the combinatorics underlying  
689 this infinite hierarchy is fascinating, and we hope to discuss this in a subsequent paper [2]. But for this paper,  
690 we will only state the Capparelli theorem in the higher case  $r = 4$  ( $m = r + 1 = 5$ ), which we can prove  
691 combinatorially by the method given in Alladi–Andrews–Gordon [5].

692 **Theorem C5** Let  $A(n)$  denote the number of partitions of  $n$  into distinct parts  $\equiv 0, 2, 4, 5, 6$ , or 8  $\pmod{10}$ .  
693 Let  $C(n)$  denote the number of partitions of  $n$  into parts  $\equiv 2, 5, 6, 14, 15$ , or 18  $\pmod{20}$ .

694 Let  $D(n)$  denote the number of partitions  $\lambda = \lambda_1 + \dots + \lambda_\ell$ , written in non-increasing order, with distinct  
695 parts not equal to 1 or 3, such that:

696 (i) The gap is  $\geq 5$  if two consecutive parts have different parities or one of them is a multiple of 5, with gap  
697 5, only allowed for consecutive multiples of 5.  
698  
699 (ii) Whenever  $5|\lambda_i$ , we have  $\lambda_i - \lambda_{i+j} \geq 5j$  for  $1 \leq i \leq i + j \leq \ell$ .

700 Then,

$$701 \quad A(n) = C(n) = D(n).$$



703 **Remark 8.4** In [2], we will consider a general form of the second condition for  $D(n)$ . Namely, for modulus  $m$   
 704 we consider partitions with the property that whenever  $m|\lambda_i$ , we have  $\lambda_i - \lambda_{i+j} \geq mj$  for  $1 \leq i \leq i+j \leq \ell$ .  
 705 We will also consider a dual variant of this notion. These considerations will be in the context of higher analogs  
 706 of  $D(n)$  featured in Theorem C5. Conversely, for lower  $m$ , namely for Capparalli (i.e.  $m = 3$ ), we note that  
 707 condition (ii) is redundant.

708 The combinatorial proof actually gives a four-parameter refinement of the equality  $A(n) = D(n)$  in which  
 709 we can keep track of the parts  $\equiv 1, 2, 3, 4 \pmod{5}$ . Note that for the partition function  $A(n)$  in Theorem  
 710 C5, the parts  $\equiv 1, 2, 3, 4 \pmod{5}$  are actually in the form  $6, 2, 8, 4 \pmod{10}$ . All this will be presented in a  
 711 subsequent paper along with the combinatorics of the infinite hierarchy.

712 For this manuscript, however, we will conclude with a discussion of the polynomial infinite hierarchy. In  
 713 particular, Alamoudi has recently noticed that the following polynomial (finite) identities correspond to (8.1)  
 714 and (8.2). For simplicity of the expressions, put  $N = v_1 + v_2 + \dots + v_r$ . Then, for integers  $M_1, M_2, \dots, M_r, L$ ,  
 715 we have

$$716 \sum_{v_1, \dots, v_r, k \geq 0} a_1^{v_1} q_1^{T_{v_1}} a_2^{v_2} q_2^{T_{v_2}} \cdots a_r^{v_r} q_r^{T_{v_r}} q^{T_k + kN} \begin{bmatrix} M_1 \\ v_1 \end{bmatrix}_{q_1} \begin{bmatrix} M_2 \\ v_2 \end{bmatrix}_{q_2} \cdots \begin{bmatrix} M_r \\ v_r \end{bmatrix}_{q_r} \begin{bmatrix} L - N \\ k \end{bmatrix}_{(-q)_N} \\ 717 = (-a_1 q_1; q_1)_{M_1} (-a_2 q_2; q_2)_{M_2} \cdots (-a_r q_r; q_r)_{M_r} (-q)_L. \quad (8.11)$$

718 Notice that when  $M_1, M_2, \dots, M_r, L$  are non-negative, with  $L \geq M_1 + M_2 + \dots + M_r$ , the sum on the left  
 719 is equivalent to the conditions that

721

$$722 \quad 0 \leq v_i \leq M_i \quad \text{for } i = 1, 2, \dots, r, \quad \text{and} \quad N + k \leq L. \quad (8.12)$$

724

To prove (8.11), sum the inner sum over  $k$  and use

$$725 \quad \sum_{k \geq 0} c^k q^{T_k + Nk} \begin{bmatrix} L - N \\ k \end{bmatrix} = (-cq^{N+1})_{L-N}, \quad (8.13)$$

727 and

$$728 \quad (-a_i q_i; q_i)_{M_i} = \sum_{v_i \geq 0} a_i q_i^{T_{v_i}} \begin{bmatrix} M_i \\ v \end{bmatrix}_{q_i} \quad \text{for } i = 1, 2, \dots, r. \quad (8.14)$$

730

When  $q_1 = \dots = q_r = q^2$ , we obtain

$$731 \quad \sum_{v_1, \dots, v_r \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{2(T_{v_1} + T_{v_2} + \dots + T_{v_r}) + T_k + Nk} \begin{bmatrix} M_1 \\ v_1 \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ v_2 \end{bmatrix}_{q^2} \cdots \begin{bmatrix} M_r \\ v_r \end{bmatrix}_{q^2} \begin{bmatrix} L - N \\ k \end{bmatrix}_{(-q)_N} \\ 733 = (-a_1 q^2; q^2)_{M_1} (-a_2 q^2; q^2)_{M_2} \cdots (-a_r q^2; q^2)_{M_r} (-q)_L. \quad (8.15)$$

734 Moreover, when  $M_i$  for  $i = 1, 2, \dots, r$  and  $L$  all  $\rightarrow \infty$ , (8.11) reduces to (8.2). On the other hand, for a  
 735 polynomial analog of (8.1) maintaining the form of (7.1), we first write the LHS of (8.1) in the form of (7.1)  
 736 to get

$$737 \quad \sum_{v_1, v_2, \dots, v_r, k \geq 0} \frac{a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{2T_{v_1} + 2T_{v_2} + \dots + 2T_{v_r} + T_k + k(v_1 + v_2 + \dots + v_r)}}{(q)_{v_1 + v_2 + \dots + v_r + k}} \begin{bmatrix} v_1 + \dots + v_r \\ v_1, \dots, v_r \end{bmatrix}_{q^2} \\ 738 \quad \times \begin{bmatrix} v_1 + \dots + v_r + k \\ v_1 + \dots + v_r, k \end{bmatrix}_q \\ 739 \quad = \sum_{v_1, v_2, \dots, v_r \geq 0} \frac{a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2T_{v_1} + 2T_{v_2} + \dots + 2T_{v_r}} (-q)_{v_1 + v_2 + \dots + v_r} (-cq^{v_1 + v_2 + \dots + v_r + 1})_\infty}{(q^2; q^2)_{v_1} (q^2; q^2)_{v_2} \cdots (q^2; q^2)_{v_r}}. \quad (8.1b)$$

741 As before, for simplicity of the expressions, put  $N = v_1 + v_2 + \dots + v_r$ . Then

$$\begin{aligned}
 742 \quad & \sum_{v_1, v_2, \dots, v_r, k \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{2(T_{v_1} + T_{v_2} + \dots + T_{v_r}) + T_k + k(v_1 + v_2 + \dots + v_r)} \frac{(q^{L-N-k+1})_k}{(q)_{N+k}} \begin{bmatrix} N \\ v_1, \dots, v_r \end{bmatrix}_{q^2} \\
 743 \quad & \times \begin{bmatrix} N+k \\ N, k \end{bmatrix}_q \prod_{i=1}^r (q^{2M_i-2v_i+2}; q^2)_{v_i} \\
 744 \quad = \sum_{v_1, v_2, \dots, v_r \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2(T_{v_1} + T_{v_2} + \dots + T_{v_r})} \begin{bmatrix} M_1 \\ v_1 \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ v_2 \end{bmatrix}_{q^2} \cdots \begin{bmatrix} M_r \\ v_r \end{bmatrix}_{q^2} (-q)_N (-cq^{N+1})_{L-N}.
 \end{aligned} \tag{8.16}$$

746 In particular, for non-negative integers  $M_1, \dots, M_r, L$ , with  $L \geq M_1 + \dots + M_r$  we have

$$\begin{aligned}
 748 \quad & \sum_{\substack{v_1, v_2, \dots, v_r, k \geq 0 \\ v_1 \leq M_1, \dots, v_r \leq M_r, N+k \leq L}} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{2(\sum_{i=1}^r T_{v_i}) + T_k + k(v_1 + v_2 + \dots + v_r)} \begin{bmatrix} L \\ N+k \end{bmatrix}_q \begin{bmatrix} N \\ v_1, \dots, v_r \end{bmatrix}_{q^2} \\
 749 \quad & \times \begin{bmatrix} N+k \\ N, k \end{bmatrix}_q \frac{\prod_{i=1}^r (q^{2M_i-2v_i+2}; q^2)_{v_i}}{(q^{L-N+1})_N} \\
 750 \quad = \sum_{v_1=0}^{M_1} \sum_{v_2=0}^{M_2} \cdots \sum_{v_r=0}^{M_r} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2(\sum_{i=1}^r T_{v_i})} \begin{bmatrix} M_1 \\ v_1 \end{bmatrix}_{q^2} \begin{bmatrix} M_2 \\ v_2 \end{bmatrix}_{q^2} \cdots \begin{bmatrix} M_r \\ v_r \end{bmatrix}_{q^2} (-q)_N (-cq^{N+1})_{L-N}.
 \end{aligned} \tag{8.17}$$

751 In the same way as before, when  $M_i$  for  $i = 1, 2, \dots, r$ , and  $L$  all  $\rightarrow \infty$ , (8.16) and (8.17) give (8.1b). We  
 752 note that just like (7.10), the LHS of (8.16) could be written in the form of the LHS of (8.11) or (8.15), then the  
 753 identity is evident. However, we emphasize this form because it maintains its resemblance to the original (7.1).  
 754 Furthermore, the form<sup>4</sup> of (8.11) does not exploit the pivotal role the substitution  $q_1 = \dots = q_r = q^2$  plays in  
 755 giving this form. For example, for the substitution  $q_1 = \dots = q_r = q^3$  it is not true that the form is preserved.  
 756 In particular, for any function  $\epsilon(q, M_1, \dots, M_r, L)$  such that  $\epsilon(q, M_1, \dots, M_r, L) \rightarrow 1$  as  $M_1, \dots, M_r, L$  all  
 757  $\rightarrow \infty$  we have

$$\begin{aligned}
 760 \quad & \sum_{v_1, v_2, \dots, v_r, k \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} c^k q^{3(T_{v_1} + T_{v_2} + \dots + T_{v_r}) + T_k + k(v_1 + v_2 + \dots + v_r)} \begin{bmatrix} L \\ N+k \end{bmatrix}_q \begin{bmatrix} N \\ v_1, \dots, v_r \end{bmatrix}_{q^3} \\
 761 \quad & \times \begin{bmatrix} N+k \\ N, k \end{bmatrix}_q \epsilon(q, M_1, \dots, M_r, L) \\
 762 \quad \neq \sum_{v_1, v_2, \dots, v_r \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{3(T_{v_1} + T_{v_2} + \dots + T_{v_r})} \begin{bmatrix} M_1 \\ v_1 \end{bmatrix}_{q^3} \begin{bmatrix} M_2 \\ v_2 \end{bmatrix}_{q^3} \cdots \begin{bmatrix} M_r \\ v_r \end{bmatrix}_{q^3} (-q)_N (-cq^{N+1})_{L-N}.
 \end{aligned}$$

764 This is because, if we take  $a_2 = \dots = a_r = c = 0$  and let  $M_1 \rightarrow \infty$ , the coefficient of  $a_1$  in LHS is  $\frac{q^3}{1-q}$  and  
 765 the coefficient of  $a_1$  in RHS is  $\frac{q^3(1+q)}{1-q^3}$ . This is to say, if we set  $q_1 = \dots = q_r = q^{m'}$ , only the case  $m' = 2$   
 766 gives the form of (7.1).

767 *Remark 8.5* Like (7.10), using the bound shifting method on (8.17) gives, for any  $L \in \mathbb{Z}$ ,

$$\begin{aligned}
 768 \quad & \sum_{v_1, \dots, v_r, k \geq 0} a_1^{v_1} \cdots a_r^{v_r} c^k q^{2(\sum_{i=1}^r T_{v_i}) + T_k + kN} \begin{bmatrix} L \\ N+k \end{bmatrix}_q \begin{bmatrix} N \\ v_1, \dots, v_r \end{bmatrix}_{q^2} \begin{bmatrix} N+k \\ N, k \end{bmatrix}_q (-q^{L-N+1})_N \\
 769 \quad = \sum_{v_1, v_2, \dots, v_r \geq 0} a_1^{v_1} a_2^{v_2} \cdots a_r^{v_r} q^{2(\sum_{i=1}^r T_{v_i})} \prod_{i=1}^r \begin{bmatrix} L - \sum_{i < j \leq r} v_j \\ v_i \end{bmatrix}_{q^2} (-q)_N (-cq^{N+1})_{L-N}.
 \end{aligned} \tag{8.18}$$

<sup>4</sup> See Remark 8.6.



771 **Remark 8.6** In [2] we will consider [5, Eq. (5.5)] as an instance of the more general equation below.

772

$$H(v_1, \dots, v_r, k) = q^{2T_{v_1} + \dots + 2T_{v_r} + T_k + k(v_1 + v_2 + \dots + v_r)} \begin{bmatrix} v_1 + \dots + v_r \\ v_1, \dots, v_r \end{bmatrix}_{q^2} \times \begin{bmatrix} v_1 + \dots + v_r + k \\ v_1 + \dots + v_r, k \end{bmatrix}_q$$

773 We will interpret the above function combinatorially in a manner that has similarities and differences to the  
774 interpretation in [5]. This interpretation highlights the combinatorial significance of the forms (7.1) and (8.1b).

775 **Work of Berkovich–Uncu:** Berkovich and Uncu [16–19], provided three separate finite polynomial identities  
776 that imply Capparelli’s identities as limiting cases. The original polynomial identities were found by imposing  
777 bounds on the combinatorial constructions that Kurşungöz [27] introduced. Their proofs used an automated  
778 deduction method and recurrences. The proven polynomial identities also led to the discovery of infinite  
779 hierarchies of sum-product identities that include Capparelli’s identities [16, 18]. Similarly, Uncu [32], by  
780 refining Kurşungöz’s construction, proved a polynomial identity that implies Schur’s partition theorem. The  
781 results and methods in this paper are different from these related works.

782 **Concluding thought:** Two things have been accomplished in this paper: (i) a unification of Schur’s theorem and  
783 Lebesgue identity, and (ii) an infinite hierarchy of  $q$ -hypergeometric identities of which the initial ones are the  
784 identities for the partition theorems of Lebesgue and Capparelli. The unification of the Schur and Lebesgue  
785 partition theorems was motivated by the fact that the basic idea behind both of their combinatorial proofs was  
786 the same—namely, a method of Bressoud suitably adapted to each of the two. Since the combinatorial proof  
787 of Capparelli’s theorem in [5] also uses the technique of Bressoud, it is our desire to unify the theorems of  
788 Schur, Lebesgue, and Capparelli by a common scheme.

789 **Acknowledgements** We thank George Andrews, Alexander Berkovich, and Ali Uncu for stimulating discussions and helpful  
790 remarks. We also appreciate the help of Aritram Dhar, who converted our amstex file to LaTeX and made pointed remarks on  
791 Theorem C5. Additionally, we are grateful to Ole Warnaar for his comments and suggestions. Lastly, we thank both referees for  
792 their helpful remarks. In particular, we thank Referee 2 for a careful reading of, and observations on, the statement of Theorem  
793 C5. This ultimately put us on the path of refining the statement.

794 **Funding** There was no funding to report.

795 **Declarations**

796 **Conflict of interest** We have no relevant financial or non-financial interest to disclose. Thus, there is no conflict of interest or  
797 competing interests.

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