SCHMIDT-TYPE THEOREMS VIA WEIGHTED PARTITION IDENTITIES

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Dedicated to the memory of Professor Richard Askey

Abstract: *A 1999 theorem of F. Schmidt states that the number of partitions into distinct parts such that the odd indexed parts sum to n, is equal to the number of partitions of n. Recently, using MacMahon's partition analysis, Andrews and Paule established two further theorems of the Schmidt-type. Here we show that Schmidt's 1999 theorem is equivalent to a weighted partition identity involving Rogers-Ramanujan partitions that I established in 1997. Using the weighted partition approach, we shall also establish combinatorially the two recent Schmidt-type theorems of Andrews-Paule. We conclude by proving another Schmidt-type theorem using weighted partitions.*

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§0: Introduction and statement of results

In 1999, F. Schmidt stated the following theorem in the American Mathematical Monthly [10] and asked for proofs:

Theorem 1 (Schmidt): *The number of partitions into distinct parts such that the odd indexed parts sum to n, is equal to the number of partitions of n.*

Proofs were given by several mathematicians including Schmidt. Here we first show that Theorem 1 is equivalent to the following weighted partition theorem in my 1997 paper [1, Theorem 1]:

Theorem 1*: *Let $\rho : b_1 + b_2 + \cdots + b_k$ be a Rogers-Ramanujan partition, namely a partition with difference ≥ 2 between parts. Put $b_{k+1} = -1$. Define the weight $w(\rho)$ as:*

$$w(\rho) = \prod_{i=1}^{k} (b_i - b_{i+1} - 1). \tag{1}$$

Then

$$\sum_{\rho \in \mathcal{R}, \sigma(\rho)=n} w(\rho) = p(n), \tag{2}$$

where $\mathcal{R}$ is the set of partitions with difference ≥ 2 between parts, and $p(n)$ is the number of (unrestricted) partitions of n.

In Theorem 1* and in what follows, $\sigma(\pi)$ denotes the sum of the parts of a partition π, namely, the integer being partitioned.

Of the different proofs of Theorem 1, the proof by Uncu [12] establishes the equivalence of Theorem 1 and 1*; his proof is related to, but different from what we give here.

I noticed Theorem 1* in 1994 during a visit to The Pennsylvania State University, and that is what led me to a study of weighted partition identities [1], [2]. In §1, the equivalence of Theorems 1 and 1* will be established and the combinatorial proof of Theorem 1* in [1] will be recalled.

By a Schmidt-type partition theorem, we mean a result involving the number $\psi(n)$ of partitions $\pi : a_1 + a_2 + \cdots$ such that a certain sub-sum of π is n. Using MacMahon's Partition Analysis, Andrews and Paule [3] recently established the following two Schmidt-type results:

Theorem 2 (Andrews-Paule): *Let $s(n)$ denote the number of partitions*

$$a_1 + a_2 + a_3 + a_4 + \cdots \tag{3}$$

satisfying

$$a_1 \geq a_2 \geq a_3 \geq a_4 \cdots \tag{4}$$

such that

$$a_1 + a_3 + a_5 + \cdots = n. \tag{5}$$

Then

$$s(n) = p_2(n), \tag{6}$$

where $p_2(n)$ is the number of partitions of n in two colors.

Remark: It has been brought to our attention that Theorem 2 was established in 2018 by Uncu [12]. But our proof (given below) is different.

Theorem 3 (Andrews-Paule): *Let $u(n, k)$ denote the number of partitions*

$$a_1 + a_2 + a_3 + \cdots + a_{3k} \tag{7}$$

satisfying

$$a_1 > a_2 > a_3 \cdots > a_{3k-1} > a_{3k} \geq 0 \tag{8}$$

such that

$$a_1 + a_4 + a_7 + \cdots + a_{3k-2} = n. \tag{9}$$

Let $v(n, k)$ denote the number of partitions of n in three colors, such there are exactly k parts of the first color with difference at least 2 between parts, exactly k parts of the second color all distinct, and at most k parts of the third color. Then

$$u(n, k) = v(n, k).$$

In §2, we will provide a new proof of Theorem 2 using weighted partitions. We also give a new proof of Theorem 3, but for this we need to reformulate it as the following weighted partition theorem:

Theorem 3*: *Let $D_3(n,k)$ denote the set of partitions*

$$\pi : b_1 + b_2 + \cdots + b_k, \quad \text{with} \quad \sigma(\pi) = n,$$

with difference ≥ 3 between parts, and with smallest part ≥ 2. Let $b_{k+1} = -1$. Define the weight $w(\pi)$ as follows:

$$w(\pi) = \prod_{i=1}^{k} \binom{b_i - b_{i+1} - 1}{2}. \tag{10}$$

Then

$$\sum_{\pi \in D_3(n,k)} w(\pi) = u(n,k) = v(n,k). \tag{11}$$

In §3 we prove the first equality in (11) and also the equality that the sum of the weights in (11) equals $v(n,k)$. Thus the proof of Theorem 3* in Section 3 is a new combinatorial proof of Theorem 3.

Guided by the combinatorial arguments in the proof of Theorem 3*, we state and prove a new Schmidt type theorem and its weighted partition version (Theorem 4) in §4. The q-hypergeometric version of Theorem 4 has an integresting history going back to the letter on January 1920 that Ramanujan wrote to Hardy (see [4], p. 220) announcing his discovery of the mock-theta functions.

Finally in §5, we allude to the work of Bowman [5] and Eichhorn [6], related to our initial research on weighted partition identities. But these weighted partition versions of the Schmidt type theorems in this paper, are different from the partitions with number in their gaps considered by Bowman and Eichhorn.

§1: Equivalence of Theorems 1 and 1*

Let $S(n)$ denote the number of partitions

$$\psi : a_1 + a_2 + a_3 + \cdots , \tag{12}$$

such that

$$a_1 > a_2 > a_3 \cdots \tag{13}$$

and

$$a_1 + a_3 + a_5 + \cdots = n. \tag{14}$$

First we show that

$$\sum_{\rho \in \mathcal{R}, \sigma(\rho) = n} w(\rho) = S(n), \tag{15}$$

with $\mathcal{R}$ as in Theorem 1*, and $w(\rho)$ as in (1). After that, we will recall the combinatorial proof in [1] that

$$\sum_{\rho \in \mathcal{R}, \sigma(\rho)=n} w(\rho) = p(n). \tag{16}$$

That will establish that Theorem 1 and 1* are equivalent, and yield a combinatorial proof of the theorems as well.

First note that the inequalities in (13) imply that $a_1, a_3, a_5 \cdots$ are integers with difference at least 2 between them. So consider a partition $\rho \in \mathcal{R}$ with $\sigma(\rho) = n$, namely

$$\rho : b_1 + b_2 + b_3 \cdots + b_k = n,$$

with $b_{k+1} = -1$, and make the following identification:

$$b_1 = a_1,\ b_2 = a_3,\ b_3 = a_5,\ \cdots .$$

In order to get all partitions enumerated by $S(n)$ given by

$$a_1 + a_3 + a_5 + \cdots + a_{2k-1} = b_1 + b_2 + b_3 + \cdots + b_k = n,$$

we need to insert a_2 in the open interval (b_1, b_2), a_4 in the open interval (b_2, b_3), and so on. The number of choices for a_2 is $b_1 - b_2 - 1$, for a_4 is $b_2 - b_3 - 1$, $\cdots$, and for a_{2k-2} is $b_{k-1} - b_k - 1$. We could have a part a_{2k} or not. If there is no part a_{2k} in ψ, we set $a_{2k} = 0$. If there is a part $a_{2k} \geq 1$ in ψ, then the number of choices for this is $b_k - 1$. So together with the possible value $a_{2k} = 0$, the number of choices for a_{2k} is

$$b_k = b_k - b_{k+1} - 1,$$

because $b_{k+1} = -1$. Now the choices of the $a_2, a_4, \cdots$ are independent of each other. So each $\rho \in \mathcal{R}$ with $\sigma(\rho) = n$ spawns $w(\rho)$ partitions of the type enumerated by $S(n)$, where $w(\rho)$ is the product as in (1). So if we sum these weights $w(\rho)$ as in (15), we will get $S(n)$. This proves (15).

Now we recall the combinatorial proof of (16) in [1].

Given an unrestricted partition π enumerated by $p(n)$, consider its Ferrers graph. In the Ferrers graph, count nodes along the hooks of the graph. This yields a partition of n with difference ≥ 2 between the parts. Call this partition $\phi(\pi) = \rho$. The mapping

$$\phi : \pi \to \phi(\pi) = \rho$$

is a surjection from the set of unrestricted partitions of n to the set of Rogers-Ramanujan partitions of n. The number of parts of ρ is the number of nodes in the descending diagonal of the Durfee square of π.

In order to realize that this map is a surjection, we start with a Rogers-Ramanujan partition $\rho : b_1 + b_2 + \cdots + b_k$ into k parts. Next draw a $k \times k$ Durfee square. Then the portion π_r to the right of the Durfee square can be completed in such a way that the hook lengths of the partition π whose Ferrers graph is the Durfee square together with the portion π_r, are precisely $b_1, b_2, \cdots, b_k$. So clearly the map ϕ is a surjection. At this point we note that the π we constructed comprising only of the Durfee square and π_r, has no nodes below the Durfee square. We call a partition π whose Ferrers graph has no nodes below the Durfee square as a *primary partition*. Thus while the map ϕ is a surjection from the unrestricted partitions of n to the Rogers-Ramanujan partitions of n, it is a *bijection* between the primary partitions of n and the Rogers-Ramanujan partitions of n. So to prove (16), all we need to do is to show that each primary partition π of n, spawns $w(\rho)$ partitions enumerated by $p(n)$, where $\rho = \phi(\pi)$. This is established by the sliding operation as in [1] that we described next.

Given the Ferrers graph of a primary partition with a $k \times k$ Durfee square, the a count of the number of nodes along its hooks yields a Rogers-Ramanujan partition ρ : $b_1 + b_2 + \cdots + b_k$, namely a partition with difference at least 2 between parts. Let the portion to the right of the Durfee square be denoted by π_r. Consider any column of π_r. By a *sliding operation*, we mean the movement of a column of π_r and the placement of this column as a row below the Durfee square. The following are invariant under a sliding operation:

(i) The size of the Durfee square.

(ii) The total number of nodes, namely the integer being partitioned.

(iii) The sizes of the hook lengths, which will be $b_1, b_2, \cdots, b_k$. Thus the partition ρ obtained by counting nodes along hooks remains unchanged.

Next note that given two consecutive parts b_i and b_{i+1}, for $1 \leq i \leq k-1$ in ρ counted as hook lengths of the primary partition, the number of columns of length i is $b_i - b_{i+1} - 2 \geq 0$. Given these columns of length i (of which there could none for a certain i), we could move $0, 1, 2, \cdots, b_i - b_{i+1} - 2$ columns and place them below the Durfee square. Thus we can perform $b_i - b_{i+1} - 1$ sliding operations on the columns of length i. For columns of length k, we can perform $b_k = b_k - b_{k+1} - 1$ sliding operations with $b_{k+1} = -1$. So a total of $w(\rho)$ sliding operations can be performed on each such ρ to generate unrestricted partitions of n, where these Ferrers graphs are now to be read row-wise in the count of $p(n)$. Thus each ρ spawns $w(\rho)$ partitions enumerated by $p(n)$. Summing these weights as in (16) yields $p(n)$. That provides a combinatorial proof of Theorem 1* and of Theorem 1 which are equivalent.

REMARK: Since we noted that ϕ is a surjection, we now point out that the size of the inverse image $|\phi^{-1}(\rho)| = w(\rho)$, for each Rogers-Ramanujan partition ρ.

§2: Proof of Theorem 2 using weighted partitions

Let $s(n)$ be as in Theorem 2. Since the a_i form a non-increasing sequence, $a_1 + a_3 + a_5 + \cdots$ is an ordinary partition of n into non-increasing parts. We now consider $\pi : b_1 + b_2 + b_3 + \cdots + b_k = n$, a partition of n and set $b_{k+1} = 0$. We make the following identifications:

$$a_1 = b_1,\ a_3 = b_2,\ \cdots,\ a_{2k-1} = b_k \geq 1. \tag{17}$$

We can choose a_2 to be any integer in the closed interval $[b_1, b_2]$. So the number of choices for a_2 is $b_1 - b_2 + 1$. Similarly, the number of choices for a_4 is $b_2 - b_3 + 1$, $\cdots$, and the number of choices for a_{2k-2} is $b_{k-1} - b_k + 1$. Finally, with regard to a_{2k}, it may be a part of a partition counted by $s(n)$ if it is ≥ 1, or not, in which case we can put $a_{2k} = 0$. If $a_{2k} \geq 1$, then it must be $\leq b_k$. So the number of choices for a_{2k} including the value 0 is $b_k + 1 = b_k - b_{k+1} + 1$. Thus each partition π of n spawns

$$w_1(\pi) = \prod_{i=1}^{k} (b_i - b_{i+1} + 1) \tag{18}$$

partitions enumerated by $s(n)$, and all partitions enumerated by $s(n)$ can be obtained in this fashion from the partitions of n. Thus we have

$$\sum_{\pi, \sigma(\pi)=n} w_1(\pi) = s(n). \tag{19}$$

Next we connect $p(n)$ with $p_2(n)$ through a weighted identity.

Given a partition $\pi : b_1 + b_2 + \cdots + b_k = n$, rewrite it as

$$b_1^* f_1 + b_2^* f_2 + \cdots + b_\nu^* f_\nu = n, \tag{20}$$

where the b_j^* are strictly decreasing positive integers, and each b_j^* occurs with frequency $f_j \geq 1$. Now let red and blue be two colors on the positive integers with the convention that blue>red for any given integer n. Given b_j^* occurring f_j times, we can color these b_j^* as follows: either all have color red, or the first has color blue followed by the rest in color red, or the first two are in color blue followed by the rest in color red, $\cdots$, or all in color blue. So the number of ways to color these b_j^* in two colors is $f_j + 1$. Thus the number of ways to two-color the given partition π is

$$w_2(\pi) = \prod_{j=1}^{\nu} (f_j + 1). \tag{21}$$

Thus each partition π of n spawns $w_2(\pi)$ partitions enumerated by $p_2(n)$, and all partitions counted by $p_2(n)$ can be obtained in this way. This yields the weighted partition identity

$$\sum_{\pi, \sigma(\pi)=n} w_2(\pi) = p_2(n). \tag{22}$$

The equality $s(n) = p_2(n)$ will follow from (19) and (22) if we show

$$\sum_{\pi, \sigma(\pi)=n} w_1(\pi) = \sum_{\pi, \sigma(\pi)=n} w_2(\pi). \tag{23}$$

We prove (23) by establishing that

$$w_1(\pi) = w_2(\pi^*), \tag{24}$$

where π^* is the conjugate of π given by the Ferrers graph of π.

Consider a partition $\pi : b_1 + b_2 + \cdots + b_k = n$ and its Ferrers graph. Rewrite this as $b_1^* f_1 + b_2^* f_2 + \cdots + b_\nu^* f_\nu = n$. So in the Ferrers graph of the partition π, there are $b_1^* - b_2^*$ columns of length f_1, $b_2^* - b_3^*$ columns of length $f_1 + f_2$, $b_3^* - b_4^*$ columns of length $f_1 + f_2 + f_3$, and so on. Think of f_1, $f_1 + f_2$, $f_1 + f_2 + f_3$, $\cdots$ as the distinct parts of π^* occurring with frequency $b_1^* - b_2^*$, $b_2^* - b_3^*$, $\cdots$. Thus the weight $w_2(\pi^*)$ as per (21) would be

$$w_2(\pi^*) = \prod_{i=1}^{\nu}(b_j^* - b_{j+1}^* + 1) = \prod_{i=1}^{k}(b_i - b_{i+1} + 1) = w(\pi), \tag{25}$$

because for the products in (25), when $b_i = b_{i+1}$, that is when a part repeats, we trivially have $b_1 - b_{i+1} + 1 = 1$ as the factor! So only *distinct* values of the b_i, namely the b_j^* contribute to the weight. Thus from (25) we see that (24) holds and hence (23). This completes the combinatorial proof of Theorem 2.

§3: Proof of Theorem 3 via weighted partitions

We have reformulated Theorem 3 as Theorem 3*. We begin by establishing the first equality in Theorem 3*.

Since the parts in the partition in (8) are strictly decreasing, it follows that the parts for the partition of n given by

$$a_1 + a_4 + a_7 + \cdots + a_{3k-2} = n$$

have difference at least 3 between parts, and $a_{3k-2} \geq 2$. So we consider the set $D_3(n,k)$ of partitions

$$\pi : b_1 + b_2 + \cdots + b_k = n$$

of n into parts that differ by ≥ 3 and with smallest part $b_k \geq 2$. We make the following identifications:

$$b_1 = a_1,\ b_2 = a_4,\ b_3 = a_7, \cdots, b_k = a_{3k-2}.$$

Now (7) tells us that we need to choose two distinct integers a_2, and a_3 in the open interval $(a_1, a_4) = (b_1, b_2)$. So the number of choices for the pair a_2, a_3 is $\binom{b_1 - b_2 - 1}{2}$. More generally, for $1 \leq i \leq k-1$, the number of choices for a_{3i-1}, a_{3i} in the open interval (b_i, b_{i+1}) is $\binom{b_i - b_{i+1} - 1}{2}$. Finally, the number of choices for a_{3k-1}, a_{3k}, in the half-open interval $(b_k, 0]$ is $\binom{b_k}{2} = \binom{b_k - b_{k+1} - 1}{2}$, because $b_{k+1} = -1$. So each partition $\pi \in D_3(n,k)$ generates $w(\pi)$ partitions enumerated by $u(n,k)$, with $w(\pi)$ as in (10). Thus summing these weights $w(\pi)$ over all $\pi \in D_3(n,k)$ yields $u(n,k)$ which proves the first equality in (11).

The proof that the sum of the weights in (11) equals $v(n,k)$ is more complicated, and we provide this now.

We think of each partition counted by $v(n,k)$ as a triple (vector)-partition (π_2, π_1, π_0), where π_2 is a partition into exactly k parts that differ by ≥ 2, π_1 is a partition into exactly

k parts that differ by ≥ 1 (namely, distinct parts), and π_0 is a partition into at most k parts. Note that π_0 is an ordinary partition and so the gaps between its parts is ≥ 0.

Observe that the conjugate π_1^* of π_1 is a partition where the set of parts is $1, 2, \cdots, k$, and these parts may repeat. Likewise, the conjugate π_2^* of π_2, is a partition where the set of parts is $1, 2, \cdots, k$, where each part with the possible exception of k occurs at least twice, but k has to occur at least once. Finally, the conjugate π_0^* of π_0 is a partition into parts $\leq k$.

Consider now a partition $\pi \in D_3(n, k)$, and its Ferrers graph. Let $\pi : b_1+b_2+\cdots+b_k = n$. For $1 \leq i \leq k-1$, since $b_i - b_{i+1} \geq 3$, there are at least 3 columns of length i in the Ferrers graph of π. Since $b_k \geq 2$, there are at least two columns of length k in the Ferrers graph of π. For each i satisfying $1 \leq i \leq k-1$, extract all columns of length i from the Ferrers graph, and first place two columns of length i in the Ferrers graph of π_2, and one column of length i in the Ferrers graph of π_1. When $i = k$, the variation is that we first place one column of length k in the Ferrers graph of π_2, and one column of length k in the Ferrers graph of π_1. In any case, for $1 \leq i \leq k$, we are left with $b_i - b_{i+1} - 3$ columns of length i to distribute as columns of π_2, π_1, and π_0 (using $b_{k+1} = -1$). If we do not put any column of length i in the graph of π_0, then we need to distribute the $b_i - b_{i+1} - 3$ columns of length i between the graphs of π_2 and π_1, giving $b_i - b_{i+1} - 2$ ways of doing so. If we place exactly one column of length i in the graph of π_0, then we have to distribute $b_i - b_{i+1} - 4$ columns of length i between the graphs of π_2 and π_1, giving the number of choices as $b_i - b_{i+1} - 3$. Next we consider what happens if we place two columns of length i in π_0, and so on. So this argument shows that the number of choices to distribute the $b_i - b_{i+1} - 3$ columns of length i between the Ferrers graphs of π_2, π_1, and π_0 is

$$(26) \qquad (b_i - b_{i+1} - 2) + (b_i - b_{i+1} - 3) + (b_i - b_{i+1} - 4) + \cdots + 1 = \binom{b_i - b_{i+1} - 1}{2}.$$

Since the distribution of columns of different lengths are "independent" of each other, we need to take the product of the binomial coefficients in (26), to get the total number of choices. This product is the weight $w(\pi)$ in (10). So each partition $\pi \in D_3(n, k)$ spawns $w(\pi)$ triple-partitions (π_2, π_1, π_0), and all such triple-partitions can be generated in this fashion. Thus summing the weights $w(\pi)$ over all $\pi \in D_3(n, k)$ yields $v(n, k)$. This completes the combinatorial proof of Theorem 3*, and hence of Theorem 3.

Remark: Andrews and Paule [3] determined the generating function of $u(n, k)$ and rewrote it to show it is the same as the generating function of $v(n, k)$. More precisely, they showed

$$(27) \qquad \sum_{n=0}^{\infty} u(n, k) q^n = \frac{q^{(3k^2+k)/2}}{(q)_k^3} = \frac{q^{k^2}}{(q)_k} \times \frac{q^{k(k+1)/2}}{(q)_k} \times \frac{1}{(q)_k} = \sum_{n=0}^{\infty} v(n, k) q^n.$$

The main step in their proof is to establish the first equality in (27), from which Theorem 3 follows by the way the generating function of $u(n, k)$ is rewritten.

§4: Another Schmidt type result

The Schmidt-type partitions considered in Theorems 1, 2, 3, are either partitions into distinct parts or ordinary (unrestricted) partitions. Two classes of partitions that are "in-between" unrestricted partitions and partitions into distinct parts are partitions

$$\alpha : a_1 + a_2 + a_3 \cdots \quad \text{with} \quad a_1 > a_2 \geq a_3 > a_4 \geq a_5 \cdots , \tag{28}$$

and

$$\beta : b_1 + b_2 + b_3 \cdots \quad \text{with} \quad b_1 \geq b_2 > b_3 \geq b_4 > b_5 \cdots . \tag{29}$$

The generating functions of these two classes of partitions have nice series representations which then equal infinite products involving residue classes mod 20. More precisely, we have two beautiful identities:

$$\phi_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}} = \prod_{j>0, j\equiv\pm1,\pm3,\pm4,\pm5,\pm7,\pm9 (mod\, 20)} \frac{1}{(1-q^j)}, \tag{30}$$

and

$$\phi_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} = \prod_{j>0, j\equiv\pm1,\pm2,\pm5,\pm6\pm8,\pm9 (mod\, 20)} \frac{1}{(1-q^j)}. \tag{31}$$

The sum in (30) is the generating function of the partitions α, and the sum in (31) is the generating function of the partitions β. Identities (30) and (31) are due to L. J. Rogers [9] and are equivalent to the Rogers-Ramanujan identities in an elementary way. These are identities (98) and (94) in Slater's list [11]. Rogers did not emphasize the partition theoretic significance of (30) and (31), which were first pointed out by Gordon [8] without proof. That the series in (30) and (31) are the generating functions of partitions of the type α and β, was established by Hirschhorn [7]. The partition interpretation of the products in (30) and (31) is obvious.

In view of the loveliness of (30) and (31) and their link with partitions of the type α and β, we are motivated to ask whether there are any Schmidt-type theorems involving partitions of the type α and β with odd-indexed parts adding up to n. Such a result for both $\alpha-$ and $\beta-$type partitions is given in Theorem 4.

Theorem 4: *Let $A(n,k)$ denote the number of partitions $\alpha : a_1 + a_2 + a_3 + \cdots + a_{2k}$, such that*

$$a_1 > a_2 \geq a_3 > a_4 \geq a_5 \cdots \geq a_{2k-1} > a_{2k} \geq 0, \tag{32}$$

and with

$$a_1 + a_3 + a_5 + \cdots = n. \tag{33}$$

Let $B(n,k)$ denote the number of partitions $\beta : b_1 + b_2 + b_3 + \cdots + b_{2k}$, such that

$$b_1 \geq b_2 > b_3 \geq b_4 > b_5 \cdots > b_{2k-1} \geq b_{2k} > 0, \tag{34}$$

and with

$$b_1 + b_3 + b_5 + \cdots = n. \tag{35}$$

Let $D(n,k)$ denote the set of partitions

$$\delta : d_1 + d_2 + d_3 + \cdots + d_k \tag{36}$$

of n into distinct parts d_j. Put $d_{k+1} = 0$. Let the weight $w(\delta)$ of the partition δ be defined as

$$w(\delta) = \prod_{j=0}^{k}(d_j - d_{j+1}). \tag{37}$$

Finally, let $V(n,k)$ denote the number of bi-partitions (π_1, π_0) of n such that π_1 is a partition into exactly k distinct parts, and π_0 is an ordinary partition into at most k parts. Then

$$A(n,k) = B(n,k) = V(n,k) = \sum_{\delta \in D(n,k)} w(\delta). \tag{38}$$

Proof: The weighted sum in (38) is the most fundamental of the four quantities there. More precisely, we will first show that the weighted sum equals $A(n,k)$. The proof that the weighted sum equals $B(n,k)$ is similar. Thus the equality $A(n,k) = B(n,k)$ follows; this equality is somewhat surprising because these two partition functions are defined by different (but similar) set of innequalities. Finally we will show that the weighted sum equals $V(n,k)$. For convenience, let us denote the weighted sum in (38) by $\sum(\delta)$.

Proof that $A(n,k) = \sum(\delta)$: Begin by observing that the inequalities in (32) imply that $a_1 + a_3 + \cdots + a_{2k-1}$ is a partition into k distinct parts. Now consider a partition δ into k distinct parts as in (36). We make the identifications $a_1 = d_1$, $a_3 = d_2$, $\cdots$, $a_{2k-1} = d_k$. To get a partition α enumerated by $A(n,k)$, we need to insert a_2 in the half-open interval $(a_1, a_3] = (d_1, d_2]$. The number of choices for a_2 is $d_1 - d_2$. Similarly, the number of choices for a_4 is $d_2 - d_3$, $\cdots$, and the number of choices for a_{2k-2} is $d_{k-1} - d_k$. Finally, the number of choices for a_{2k} is $d_k = d_k - d_{k+1}$, because $d_{k+1} = 0$. So to get the total number of partitions enumerated by $A(n,k)$ generated by a given $\delta \in D(n,k)$, we need to take the product of the $d_i - d_{i+1}$ for $1 \leq i \leq k$, and this yields the weight $w(\delta)$ in (37). Thus each $\delta \in D(n,k)$, spawns $w(\delta)$ partitions generated by $A(n,k)$, and all partitions enumerated by $A(n,k)$ can be obtained by this insertion process. Thus summing $w(\delta)$ over all $\delta \in D(n,k)$, we see that $A(n,k) = \sum(\delta)$.

Proof that $B(n,k) = \sum(\delta)$ and $A(n,k) = B(n,k)$: The proof is very similar. Here given δ as in (36), we make the identifications $b_1 = d_1$, $b_3 = d_2$, $\cdots$, $b_{2k-1} = d_k$. Here the b_2 is an integer in the half-open interval $[b_1, b_3) = [d_1, d_2)$, and so the number of choices for b_2 is again $d_1 - d_2$. Similarly, the number of choices of a_{2i} is $b_i - b_{i+1}$ for $1 \le i \le k-1$. Finally a_{2k} is to be chosen from the half-open interval $[d_k, 0)$, and the number of choices is again $d_k = d_k - d_{k+1}$. Then the above arguments will show that $B(n,k) = \sum(\delta)$. Clearly $A(n,k) = B(n,k)$ follows from this.

Proof that $V(n,k) = \sum(\delta)$: Consider the Ferrers graphs of the partitions π_1 and π_0 in the bi-partitions enumerated by $V(n,k)$. The conjugate partition π_1^* of π_1 is a partition for which the set of parts is $1, 2, \cdots, k$. Similarly, the conjugate π_0^* of π_0 is a partition into parts $\le k$ in size.

Now consider a partition $\delta \in D(n,k)$, and given by (36). The Ferrers graph of δ, consists precisely of $d_i - d_{i+1}$ columns of length i, for $1 \le i \le k$, using the definition $d_{k+1} = 0$. For each i, we take out all the $d_i - d_{i+1}$ columns of length i from δ, and distribute them as columns of length i in π_1 and π_0. First we place one column of length i in the Ferrers graph of π_1. So we are left with $d_i - d_{i+1} - 1$ columns of lenght i to be distributed between the graphs of π_1 and π_0. The number of choices of doing this is $d_i - d_{i+1}$. Since the choices for the different i are independent, we multiply the $d_i - d_{i+1}$ for $1 \le i \le k$, and get the weight $w(\delta)$. Thus each $\delta \in D(n,k)$ spawns $w(\delta)$ bi-partitions enumerated by $V(n,k)$. Thus summing these weights over all $\delta \in D(n,k)$ yields $V(n,k)$. This proves that $V(n,k) = \sum(\delta)$. This completes the proof of Theorem 4.

The Generating function: The advantage of introducing $V(n,k)$ in Theorem 4 is that its generating function can be written down immediately, namely

$$\sum_{n=0}^{\infty} V(n,k) q^n = \frac{q^{k(k+1)/2}}{(q)_k} \times \frac{1}{(q)_k} = \frac{q^{k(k+1)/2}}{(q)_k^2}. \tag{39}$$

Thus

$$f(z;q) := \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} z^k V(n,k) = \sum_{k=0}^{\infty} \frac{z^k q^{k(k+1)/2}}{(q)_k^2}. \tag{40}$$

I asked George Andrews whether the infinite sum in (40) has been investigated and whether there is something of special interest about it. He then responded saying that it has a very interesting history which I now briefly describe.

As is well known, in his last letter to Hardy dated 20 Jan, 1920, Ramanujan communicated his discovery of the mock-theta functions which he classified into orders 3, 5 and 7. In that letter he provided an example of a function naturally defined by a q-series, which is NOT a mock-theta function. The example he gave was the function given by the series in (40) with $z = 1$ (see [4], p. 220). Typically, Ramanujan never mentioned the partition or combinatorial interpretation of his identities, and that was the case with (40) as well. Even though $f(1;q)$ is not a mock-theta function, Ramanujan indicated its asymptotic behavior as $q \to 1^-$, but did not provide a proof.

In his celebrated Retiring Presidential Address [13] titled "The Final Problem - an account of the mock-theta functions", G. N. Watson analyzed Ramanujan's mock-theta functions of order 3 in detail. But in [11], he also discussed the asymptotic behavior of $f(1,q)$, but heuristically. In doing so, he wrote $f(1,q)$ as follows:

$$f(1,q) = \frac{1}{(q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q)_k} (q^{k+1})_\infty$$

$$= \frac{1}{(q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q)_k} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mk+(m(m+1)/2)}}{(q)_m}. \tag{41}$$

Now use the relation

$$\frac{k(k+1)}{2} + \frac{m(m+1)}{2} + mk = \frac{(m+k)(m+k+1)}{2}. \tag{42}$$

If we set $n = m + k$, and rewrite (41) using (42), we get

$$f(1,q) = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q. \tag{43}$$

At this stage, use the identity

$$\sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q = 0, \quad \text{if} \quad n \quad \text{is odd, and} \quad = (q;q^2)_\ell, \quad \text{if} \quad n = 2\ell \quad \text{is even,} \tag{44}$$

to rewrite the expression in (43) as

$$f(1,q) = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q^2;q^2)_n}. \tag{44}$$

The infinite sum in (44) is the generating function of partitions in which the largest part is odd, all integers not exceeding the largest part occur as parts, and only the even parts repeat. Watson did not consider the partition interpretation of $f(1,q)$. Thus Theorem 4 is new.

§5: Related work

In 1996, Bowman [5] considered "partitions with numbers in their gaps". That is, given a partition π, he would insert numbers in-between the parts of π, but in doing so, the new expression π^+ is not necessarily a partition, but a restricted composition. Starting from certain classes of partitions π, he considered the number of such π^+ whose parts would add up to n. This is different from the Schmidt-type theorems, where only the sum of the parts of the sub-partition π is n. Bowman's first result is that if one starts with Rogers-Ramanujan partitions $\pi : b_1 + b_2 + \cdots + b_k$, set $b_0 = \infty$, $b_{k+1} = -1$ (my convention, not his), and create π^+ by inserting at most $b_i - b_{i+1} - 2$ ones between b_i and b_{i+1}, for

$1 \leq i \leq k$, permitting an arbitrary number of ones before b_1, then the number of such partitions π^+ of n equals $p(n)$, the number of unrestricted partitions of n. He has other interesting results of this nature in [5]. In view of my Theorem 1 in [1], which is a weighted partition identity connecting Rogers-Ramanujan partitions with $p(n)$, I conjectured that there ought to be links between weighted partition identities and partitions with numbers in gaps. Eichhorn [6] established such links between certain weighted identies of mine in [1] and some results of Bowman in [5]. While they appear related, the partitions with numbers in their gaps are different from Schmidt-type partitions. The principal idea of this paper has been to view every class of Schmidt-type partition as a class of weighted partitions, and to use these weights to provide combinatorial proofs of the Schmidt-type theorems. This view point might lead to new Schmidt-type theorems, such as Theorem 4 of this paper.

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