REVISITING THE RIEMANN ZETA FUNCTION AT POSITIVE EVEN INTEGERS

KRISHNASWAMI ALLADI AND COLIN DEFANT

Abstract. Using Parseval's identity for the Fourier coefficients of x^k, we provide a new proof that $\zeta(2k) = \dfrac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!}$.

1. Introduction

One of the most famous mathematical problems was the evaluation of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The Italian mathematician Pietro Mengoli originally posed this problem in 1644. The problem was later popularized by the Bernoullis who lived in Basel, Switzerland, so this became known as the *Basel Problem* [3]. Leonhard Euler solved this brilliantly when he was just twenty-eight years old by showing that the sum in question was equal to $\pi^2/6$. Indeed, this was Euler's first mathematical work, and it brought him world fame. In solving the Basel Problem, Euler found closed-form evaluations more generally of

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

for all even integers $2k \geq 2$. The value of $\zeta(2k)$ is given as a rational multiple of π^{2k}. More precisely, Euler showed that [6, Page 16]

$$\zeta(2k) = \frac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!}, \tag{1}$$

where the Bernoulli numbers B_m are defined by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}.$$

Since Euler's time, numerous proofs of his formula for $\zeta(2k)$ have been given. Our goal here is to give yet another proof of (1) that involves a new identity for Bernoulli numbers and a different induction process than used previously. We consider the Fourier coefficients $a_n(k)$ and $b_n(k)$ of the function that is periodic of period 2π and is given by $f(x) = x^k$ on the interval $(-\pi, \pi]$, and we obtain a pair of intertwining recurrences for these coefficients (see (7) and (8) below). We then apply Parseval's theorem to connect these Fourier coefficients with $\zeta(2k)$ and then establish (1) by solving this recurrence. The novelty in our approach is a new identity for Bernoulli numbers that we obtain and use.

A very recent paper by Navas, Ruiz, and Varona [5] establishes new connections between the Fourier coefficients of many fundamental sequences of polynomials such as the Legendre polynomials and the Gegenbauer polynomials by making them periodic of period 1. In doing so, these authors consider the Fourier coefficients of x^k, but they do not use their ideas to establish Euler's formula (1). Another paper by Kuo [4] provides a recurrence for the values of $\zeta(2k)$ involving only the values $\zeta(2j)$ for $j \leq k/2$ instead of requiring $j \leq k-1$ as we do. Our method is very different from Kuo's. Other recent proofs of Euler's formula appear in [1] and [2].

2. Warm Up

For each positive integer k, let $a_n(k)$ and $b_n(k)$ be the n^{th} Fourier coefficients of the function $x \mapsto x^k$. That is, $a_0(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} x^k\,dx$, $a_n(k) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^k \cos(nx)\,dx$, and $b_n(k) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^k \sin(nx)\,dx$ for $n, k \geq 1$. Parseval's identity [7, page 191] informs us that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^{2k}\,dx = 2a_0(k)^2 + \sum_{n=1}^{\infty}(a_n(k)^2 + b_n(k)^2). \tag{2}$$

Let us apply (2) in the case $k = 1$. We easily calculate that $a_0(1) = a_n(1) = 0$ and $b_n(1) = 2\dfrac{(-1)^{n+1}}{n}$ for all $n \geq 1$. This implies that

$$\frac{2\pi^2}{3} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty} b_n(1)^2 = \sum_{n=1}^{\infty}\frac{4}{n^2},$$

so we obtain Euler's classic result $\zeta(2) = \dfrac{\pi^2}{6}$. In Section 4, we use Parseval's identity to obtain a new inductive proof of Euler's famous formula (1).

3. An Identity for Bernoulli Numbers

Before we proceed, let us recall some well-known properties of Bernoulli numbers (see [6, Chapter 1]). In Lemma 3.1, we also establish one new identity involving these numbers.

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_2 = \frac{1}{6}$. If $m \geq 3$ is odd, then $B_m = 0$. The equation

$$\sum_{m=0}^{n-1} B_m \binom{n}{m} = 0 \tag{3}$$

holds for any integer $n \geq 2$. If n is odd, then

$$\sum_{m=0}^{n} B_m 2^m \binom{n}{m} = 0. \tag{4}$$

One can prove (4) by evaluating the Bernoulli polynomial $B_n(x) = \sum_{\ell=0}^{n} B_\ell \binom{n}{\ell} x^{n-\ell}$ at $x = 1/2$ and using the known identity $B_n(x) = (-1)^n B_n(1-x)$.

In what follows, we write $\binom{n}{r_1, r_2, r_3}$ to denote the trinomial coefficient given by $\dfrac{n!}{r_1! r_2! r_3!}$.

Lemma 3.1. *For any positive integer k,*

$$\sum_{i+t \leq \lfloor k/2 \rfloor} B_{2t} 2^{2t} \binom{2k+2}{2t, 2i+1, 2k-2t-2i+1} = (k+1)\left(2^{2k} + (-1)^k \binom{2k}{k}\right),$$

where the sum ranges over all nonnegative integers i and t satisfying $i + t \leq \lfloor k/2 \rfloor$.

Proof. The difficulty in evaluating the given sum spawns from its unusual limits of summation. Therefore, we will first evaluate the sum obtained by allowing i and t to range over all nonnegative integers satisfying $i + t \leq k$. Because $\sum_{i=0}^{k-t} \binom{2(k-t)+2}{2i+1} = 2^{2k-2t+1}$, we have

$$\sum_{i+t \leq k} B_{2t} 2^{2t} \binom{2k+2}{2t, 2i+1, 2k-2t-2i+1} = \sum_{t=0}^{k} B_{2t} 2^{2t} \binom{2k+2}{2t} \sum_{i=0}^{k-t} \binom{2(k-t)+2}{2i+1} = 2^{2k+1} \sum_{t=0}^{k} B_{2t} \binom{2k+2}{2t}.$$

Since $B_m = 0$ for all odd $m \geq 3$,

$$\sum_{i+t \leq k} B_{2t} 2^{2t} \binom{2k+2}{2t, 2i+1, 2k-2t-2i+1} = 2^{2k+1} \left(\sum_{\ell=0}^{2k+1} B_\ell \binom{2k+2}{\ell} - B_1 \binom{2k+2}{1} \right) = 2^{2k+1}(k+1). \tag{5}$$

Note that we used (3) along with the fact that $B_1 = -\frac{1}{2}$ to deduce the last equality above.

We next compute

$$\sum_{\lfloor k/2 \rfloor < i+t \leq k} B_{2t} 2^{2t} \binom{2k+2}{2t, 2i+1, 2k-2t-2i+1} = \sum_{m=\lfloor k/2 \rfloor + 1}^{k} \binom{2k+2}{2m+1} \sum_{t=0}^{m} B_{2t} 2^{2t} \binom{2m+1}{2t}$$

$$= \sum_{m=\lfloor k/2 \rfloor + 1}^{k} \binom{2k+2}{2m+1} \left(\sum_{\ell=0}^{2m+1} B_\ell 2^\ell \binom{2m+1}{\ell} - 2B_1 \binom{2m+1}{1} \right) = \sum_{m=\lfloor k/2 \rfloor + 1}^{k} \binom{2k+2}{2m+1} (2m+1),$$

where we have used (4) to see that $\sum_{\ell=0}^{2m+1} B_\ell 2^\ell \binom{2m+1}{\ell} = 0$. Therefore,

$$\sum_{\lfloor k/2 \rfloor < i+t \leq k} B_{2t} 2^{2t} \binom{2k+2}{2t, 2i+1, 2k-2t-2i+1} = \sum_{m=\lfloor k/2 \rfloor + 1}^{k} (2k+2) \left[\binom{2k}{2m} + \binom{2k}{2m-1} \right]$$

$$= (k+1) \sum_{m=\lfloor k/2 \rfloor + 1}^{k} \left[\binom{2k}{2m} + \binom{2k}{2k-2m} + \binom{2k}{2m-1} + \binom{2k}{2k-2m+1} \right] = (k+1)\left(2^{2k} - (-1)^k \binom{2k}{k}\right). \tag{6}$$

Lemma 3.1 now follows if we subtract (6) from (5). $\square$

4. The Formula for $\zeta(2k)$

To begin this section, let us use integration by parts to see that for any $n \geq 1$ and $k \geq 2$,

$$a_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \cos(nx)\,dx = -\frac{k}{n}\frac{1}{\pi}\int_{-\pi}^{\pi} x^{k-1}\sin(nx)\,dx = -\frac{k}{n}\,b_n(k-1). \tag{7}$$

Similarly, if k is odd, then

$$b_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \sin(nx)\,dx = 2\frac{(-1)^{n+1}\pi^{k-1}}{n} + \frac{k}{n}\,a_n(k-1). \tag{8}$$

Now, it is clear from the definitions of $a_n(k)$ and $b_n(k)$ that $a_n(k) = 0$ whenever k is odd and $b_n(k) = 0$ whenever k is even. Hence, $a_n(k)^2 + b_n(k)^2$ is actually equal to $(a_n(k) + b_n(k))^2$. If we appeal to (7) and (8) recursively and use the fact that $b_n(1) = 2\frac{(-1)^{n+1}}{n}$, then a simple inductive argument shows that when $n, k \geq 1$,

$$a_n(k) + b_n(k) = \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} c_n(k,\ell)\frac{\pi^{2\ell}}{n^{k-2\ell}}, \tag{9}$$

where we define

$$c_n(k,\ell) = \begin{cases} \frac{2k!}{(2\ell+1)!}(-1)^{\lfloor k/2 \rfloor + \ell + n + 1}, & \text{if } 0 \leq \ell \leq \lfloor \frac{k-1}{2} \rfloor; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Gathering (2), (9), and (10) together yields

$$2\frac{\pi^{2k}}{2k+1} - 2a_0(k)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2k}\,dx - 2a_0(k)^2 = \sum_{n=1}^{\infty}\left(\sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} c_n(k,\ell)\frac{\pi^{2\ell}}{n^{k-2\ell}}\right)^2$$

$$= \sum_{n=1}^{\infty}\sum_{j=0}^{2\lfloor \frac{k-1}{2} \rfloor} r(k,j)\frac{\pi^{2j}}{n^{2k-2j}} = \sum_{j=0}^{2\lfloor \frac{k-1}{2} \rfloor} r(k,j)\pi^{2j}\zeta(2k-2j), \tag{11}$$

where

$$r(k,j) = \sum_{i=0}^{j} c_n(k,i)c_n(k,j-i) = \sum_{i=\max\left(0,j-\lfloor \frac{k-1}{2} \rfloor\right)}^{\min\left(\lfloor \frac{k-1}{2} \rfloor, j\right)} c_n(k,i)c_n(k,j-i)$$

$$= \sum_{i=\max\left(0,j-\lfloor \frac{k-1}{2} \rfloor\right)}^{\min\left(\lfloor \frac{k-1}{2} \rfloor, j\right)} \frac{2k!}{(2i+1)!}(-1)^{\lfloor k/2 \rfloor + i + n + 1}\frac{2k!}{(2j-2i+1)!}(-1)^{\lfloor k/2 \rfloor + j - i + n + 1}$$

$$= \frac{4(-1)^j k!^2}{(2j+2)!}\sum_{i=\max\left(0,j-\lfloor \frac{k-1}{2} \rfloor\right)}^{\min\left(\lfloor \frac{k-1}{2} \rfloor, j\right)} \binom{2j+2}{2i+1}. \tag{12}$$

Let $\mu_j(k) = \displaystyle\sum_{i=\max\left(0,j-\left\lfloor\frac{k-1}{2}\right\rfloor\right)}^{\min\left(\left\lfloor\frac{k-1}{2}\right\rfloor,j\right)} \binom{2j+2}{2i+1}$. If $j \le \left\lfloor\frac{k-1}{2}\right\rfloor$, then $\mu_j(k) = \displaystyle\sum_{i=0}^{j}\binom{2j+2}{2i+1} = 2^{2j+1}$. If $\left\lfloor\frac{k-1}{2}\right\rfloor < j \le k-1$, then

$$\mu_j(k) = 2^{2j+1} - \sum_{i=0}^{j-\left\lfloor\frac{k-1}{2}\right\rfloor-1}\binom{2j+2}{2i+1} - \sum_{i=\left\lfloor\frac{k-1}{2}\right\rfloor+1}^{j}\binom{2j+2}{2i+1} = 2^{2j+1} - 2\sum_{i=0}^{j-\left\lfloor\frac{k-1}{2}\right\rfloor-1}\binom{2j+2}{2i+1}.$$

Assume inductively that we have proven (1) when k is replaced by any smaller positive integer. Using (11), (12), and this inductive hypothesis, we find that

$$2\frac{\pi^{2k}}{2k+1} - 2a_0(k)^2 = r(k,0)\zeta(2k) + \sum_{j=1}^{2\left\lfloor\frac{k-1}{2}\right\rfloor}\frac{4(-1)^j k!^2}{(2j+2)!}\pi^{2j}\zeta(2k-2j)\mu_j(k)$$

$$= 4k!^2\zeta(2k) + \sum_{j=1}^{2\left\lfloor\frac{k-1}{2}\right\rfloor}\frac{4(-1)^j k!^2}{(2j+2)!}\pi^{2j}\frac{(-1)^{k-j+1}B_{2k-2j}(2\pi)^{2k-2j}}{2(2k-2j)!}\mu_j(k)$$

$$= 4k!^2\zeta(2k) + 2^{2k+2}(-1)^{k+1}\pi^{2k}k!^2\sum_{j=1}^{k-1}\frac{B_{2k-2j}\mu_j(k)}{2^{2j+1}(2j+2)!(2k-2j)!}$$

(changing the limits of summation in the last line above is valid because $\mu_{k-1}(k) = 0$ when k is even). Consequently,

$$\frac{2(2k)!\zeta(2k)}{(2\pi)^{2k}} = \left(\frac{2^{-2k}}{2k+1} - \frac{a_0(k)^2}{(2\pi)^{2k}}\right)\binom{2k}{k} - (-1)^{k+1}2(2k)!\sum_{j=1}^{k-1}\frac{B_{2k-2j}\mu_j(k)}{2^{2j+1}(2j+2)!(2k-2j)!}$$

$$= \left(\frac{2^{-2k}}{2k+1} - \frac{a_0(k)^2}{(2\pi)^{2k}}\right)\binom{2k}{k} + (-1)^k 2(2k)!\sum_{j=1}^{k-1}\frac{B_{2k-2j}}{(2j+2)!(2k-2j)!}$$

$$-(-1)^k 2(2k)!\sum_{j=\left\lfloor\frac{k-1}{2}\right\rfloor+1}^{k-1}\frac{B_{2k-2j}\sum_{i=0}^{j-\left\lfloor\frac{k-1}{2}\right\rfloor-1}\binom{2j+2}{2i+1}}{2^{2j}(2j+2)!(2k-2j)!}. \tag{13}$$

Invoking the identity (3), we may write

$$2(2k)!\sum_{j=1}^{k-1}\frac{B_{2k-2j}}{(2j+2)!(2k-2j)!} = \frac{2}{(2k+1)(2k+2)}\sum_{m=1}^{k-1}B_{2m}\binom{2k+2}{2m}$$

$$= \frac{2}{(2k+1)(2k+2)}\left(k - B_{2k}\binom{2k+2}{2k}\right) = \frac{2k}{(2k+1)(2k+2)} - B_{2k}. \tag{14}$$

Furthermore, we may apply Lemma 3.1 to see that

$$2(2k)!\sum_{j=\left\lfloor\frac{k-1}{2}\right\rfloor+1}^{k-1}\frac{B_{2k-2j}\sum_{i=0}^{j-\left\lfloor\frac{k-1}{2}\right\rfloor-1}\binom{2j+2}{2i+1}}{2^{2j}(2j+2)!(2k-2j)!}=2\sum_{t=1}^{\lfloor k/2\rfloor}\frac{B_{2t}\sum_{i=0}^{\lfloor k/2\rfloor-t}\binom{2k-2t+2}{2i+1}}{2^{2k-2t}(2k+1)(2k+2)}\binom{2k+2}{2t}$$

$$=\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[\sum_{i+t\le\lfloor k/2\rfloor}B_{2t}2^{2t}\binom{2k+2}{2t,2i+1,2k-2t-2i+1}-\sum_{i=0}^{\lfloor k/2\rfloor}\binom{2k+2}{2i+1}\right]$$

$$=\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[(k+1)\left(2^{2k}+(-1)^k\binom{2k}{k}\right)-\left(2^{2k}+\binom{2k+1}{k}e_k\right)\right],\tag{15}$$

where $e_k=0$ if k is odd and $e_k=1$ if k is even.

If k is odd, then (13), (14), and (15) combine to show that $\dfrac{2(2k)!\zeta(2k)}{(2\pi)^{2k}}$ is equal to

$$\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[(k+1)\binom{2k}{k}-k2^{2k}+(k+1)\left(2^{2k}-\binom{2k}{k}\right)-2^{2k}\right]+B_{2k},$$

which simplifies to B_{2k}. This yields (1) when k is odd. If k is even, $a_0(k)^2=\dfrac{\pi^{2k}}{(k+1)^2}$. Hence, we may again invoke (13), (14), and (15) when k is even to find that

$$\frac{2(2k)!\zeta(2k)}{(2\pi)^{2k}}=\left(\frac{2^{-2k}}{2k+1}-\frac{a_0(k)^2}{(2\pi)^{2k}}\right)\binom{2k}{k}+\frac{2k}{(2k+1)(2k+2)}-B_{2k}$$

$$-\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[(k+1)\left(2^{2k}+\binom{2k}{k}\right)-\left(2^{2k}+\binom{2k+1}{k}\right)\right]$$

$$=\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[\frac{k^2}{k+1}\binom{2k}{k}+k2^{2k}-(k+1)\left(2^{2k}+\binom{2k}{k}\right)+\left(2^{2k}+\binom{2k+1}{k}\right)\right]-B_{2k}$$

$$=\frac{2^{1-2k}}{(2k+1)(2k+2)}\left[\frac{k^2}{k+1}\binom{2k}{k}-(k+1)\binom{2k}{k}+\frac{2k+1}{k+1}\binom{2k}{k}\right]-B_{2k}=(-1)^{k+1}B_{2k}.$$

This proves (1) when k is even, completing the induction.

5. Acknowledgment

We thank the referee for a very careful reading of the manuscript and for helpful suggestions.

References

[1] Ó. Ciaurri, L. M. Navas, F. J. Ruiz, and J. L. Varona, A simple computation of $\zeta(2k)$. Amer. Math. Monthly, **122** (2015), 444–451.
[2] E. De Amo, M. Díaz Carrillo, and J. Fernández-Sánchez, Another proof of Euler's formula for $\zeta(2k)$. Proc. Amer. Math. Soc., **139** (2011), 1441–1444.
[3] J. Havil, Gamma: exploring Euler's constant. Princeton University Press, 2003.
[4] H. Kuo, A recurrence formula for $\zeta(2n)$. Bull. Amer. Math. Soc. **55** (1949), 573–574.

[5] L. M. Navas, F. J. Ruiz, and J. L. Varona, Old and new identities for Bernoulli polynomials via Fourier series. Int. J. Math. Math. Sci. (2012), Art. ID 129126.
[6] H. Rademacher, Topics in analytic number theory. Springer Science and Business Media, 2012.
[7] W. Rudin, Principles of mathematical analysis, third edition. New York: McGraw-Hill, 1964.