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Ramanujan's legacy: the work of the SASTRA prize winners†

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Discussion

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The SASTRA Ramanujan Prize is an annual \$10 000 prize given to mathematicians not exceeding the age of 32 for revolutionary contributions to areas influenced by Srinivasa Ramanujan. The prize has been unusually successful in recognizing highly gifted mathematicians at an early stage of their careers who have gone on to shape the development of mathematics. We describe the fundamental contributions of the winners and the impact they have had on current research. Several aspects of the work of the awardees either stem from or have been strongly influenced by Ramanujan's ideas.

This article is part of a discussion meeting issue 'Srinivasa Ramanujan: in celebration of the centenary of his election as FRS'.

1. The SASTRA Ramanujan Prize

The very mention of Ramanujan's name reminds us of the thrill of mathematical discovery and of revolutionary contributions in youth. So one of the best ways to foster the legacy of Ramanujan is to recognize brilliant young mathematicians for their fundamental contributions. The *SASTRA Ramanujan Prize* is an annual \$10 000 prize given to mathematicians not exceeding the age of 32 for path-breaking contributions to areas influenced by Srinivasa Ramanujan. The age limit of the prize has been set at 32 because Ramanujan achieved so much in his brief life of 32 years. So the challenge to the candidates is to show what they have achieved in that same time frame.

The prize is given each year around 22 December (Ramanujan's birthday) in Kumbakonam (Ramanujan's hometown) at an International Conference held at SASTRA University during 21–22 December. The prize has been unusually effective in recognizing extremely gifted mathematicians at an early stage of their careers who have gone on to make further fundamental contributions and who have been rewarded with other

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prizes with a hallowed tradition such as the Fields Medal. The SASTRA Ramanujan Prize is now considered to be one of the most prestigious and coveted mathematical prizes in the world.

(a) SASTRA University and its annual Ramanujan conferences

The Shanmugha Arts, Science, Technology and Research Academy (SASTRA) is a private university created about 35 years ago in the town of Tanjore in the state of Tamil Nadu (formerly Madras State) in South India. In 2003, SASTRA purchased Ramanujan's home in Kumbakonam in the Tanjore District. To mark that occasion, SASTRA opened a branch campus in Kumbakonam called the Srinivasa Ramanujan Centre and started annual mathematics conferences there. I have been associated with SASTRA since that first conference in 2003, when I was asked to bring a team of internationally renowned researchers.

At the 2004 conference, the Vice-Chancellor, Prof. R. Sethuraman, told me that he would like to give $10 000 annually in memory of Ramanujan and asked for my suggestions. I suggested the creation of the SASTRA Ramanujan Prize and the Vice-Chancellor immediately endorsed it. At the Vice-Chancellor's invitation, I have chaired the Prize Committee since the inception of the Prize in 2005. The phenomenal success of the Prize is because of the enthusiastic support of the international mathematical community, especially that of several mathematical luminaries who have made nominations, written letters on the work of the candidates or have served on the Prize Committee. The Prize Committee consists of six members besides me, and each of the six members serves a 2 year term. I am non-voting Chair, and will vote only to break a tie.

The SASTRA Ramanujan Prize is one of the most significant efforts worldwide to preserve and honour Ramanujan's legacy and to enhance and recognize fundamental research in areas influenced by Ramanujan. I will now briefly discuss the seminal contributions of the prize winners and highlight the aspects which relate to Ramanujan's own ideas. We describe only the work for which the prizes were given, and in some instances allude to later work of the winners. It is to be noted that some of the papers of the winners that are referenced appeared a few years after the prizes were given. Instead of listing all references at the end, we give the references relating to the work of each winner right after describing his/her work, to make it easier for the reader.

(b) The SASTRA Ramanujan Prize winners

2005: Manjul Bhargava (Princeton) and Kannan Soundararajan (Michigan)—two full prizes, not shared; **2006:** Terence Tao (UCLA); **2007:** Ben Green (Cambridge); **2008:** Akshay Venkatesh (Stanford); **2009:** Kathrin Bringmann (Cologne and Minnesota); **2010:** Wei Zhang (Harvard); **2011:** Roman Holowinsky (Ohio State); **2012:** Zhiwei Yun (MIT and Stanford); **2013:** Peter Scholze (Bonn); **2014:** James Maynard (Oxford); **2015:** Jacob Tsimerman (Toronto); **2016:** Kaisa Matomäki (Turku, Finland) and Maksym Radziwill (McGill and Rutgers); **2017:** Maryna Viazovska (EPF Lausanne); **2018:** Yifeng Liu (Yale) and Jack Thorne (Cambridge).

2. The work of the SASTRA Prize winners

(a) 2005 winner Manjul Bhargava

In 1801, Gauss discovered a composition law for binary quadratic forms. Introducing several new and unexpected ideas, Bhargava broke an impasse since the time of Gauss and established composition laws for higher degree forms in his 2001 PhD thesis at Princeton University. He applied these to solve new cases of one of the fundamental questions of number theory—that of the asymptotic enumeration of number fields of a given degree. This had far-reaching consequences for various problems in algebraic number theory, such as in determining the average size of ideal class groups. He published the results in his thesis in a series of papers in the *Annals of Mathematics* [1–5]. Bhargava's lecture at SASTRA University on 22 December 2005, upon accepting the prize, was on a different topic; he announced his joint work with Jonathan Hanke on the solution to the problem of determining all universal quadratic forms—a problem whose origin can be traced back to Ramanujan. So we describe this work now. The Bhargava–Hanke solution resolved this problem in a strong form as conjectured by Conway.

A celebrated 1770 theorem of Lagrange states:

Every positive integer is a sum of (at most) four squares.

This motivated Ramanujan to investigate those (integer coefficient) quadratic forms which would represent all positive integers with integer values of the arguments. Such quadratic forms are called *universal quadratic forms*. In his notebooks, Ramanujan wrote 54 examples of such universal quadratic forms and later published his results while at the University of Cambridge:

S. Ramanujan. 'On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$'. *Proc. Camb. Phil. Soc.* **19** (1916), 11–21.

One can think of the Lagrange quadratic form as $x^2 + y^2 + z^2 + w^2$. A few examples of Ramanujan's universal quadratic forms are:

$$x^2 + y^2 + z^2 + 2w^2, \quad x^2 + y^2 + 2z^2 + 3w^2 \quad \text{and} \quad x^2 + 2y^2 + 3z^2 + 6w^2.$$

In 1993, Conway and Schneeberger claimed (unpublished):

For a positive definite quadratic form whose matrix has all integer entries, to decide if it is universal, we need only check that it represents integers 1 thru 15.

Indeed one needs to only check that the 'critical numbers' 1, 2, 3, 5, 6, 7, 10, 14 and 15 are represented. Bhargava [6] gave a new proof of the Conway–Schneeberger claim using geometric notions. An introduction to [6] is given by Conway [7].

If one only requires that coefficients of the quadratic form be integers, but not necessarily of its matrix, then the test for universality is harder. Here we have:

Conway's conjecture: *In general, to decide if a positive-definite integer coefficient quadratic form is universal, it suffices to check that a set of 29 critical integers up to 290 are represented, namely:*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26,$$
$$29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290.$$

In 2005 Bhargava and Jonathan Hanke proved Conway's conjecture and also determined all 6436 positive definite integral quaternary (four variables) quadratic forms which are universal [8]. They used the circle method of Hardy–Ramanujan and bounds for coefficients of modular forms of integral weight due to Deligne.

Bhargava also formulated and established the following general

Finiteness theorem. *Let S be any subset of the positive integers. Then there exists a unique minimal finite subset T of S, such that a positive definite integer-coefficient quadratic form represents all members of S if and only if it represents all members of T.*

Bhargava continued to do revolutionary work, such as on the average rank of elliptic curves, and for that he was later recognized with the Fields Medal in 2014.

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(b) 2005 winner Kannan Soundararajan

Soundararajan had made brilliant contributions to many parts of analytic number theory, including mutiplicative number theory, the Riemann zeta function and Dirichlet L-functions, especially on the distribution and location of their zeros, the analytic theory of automorphic forms and the Katz–Sarnak theory of symmetric groups associated with families of L-functions. He also established deep results in random matrix theory, a subject which has fundamental connections with prime number theory. With Hugh Montgomery, he showed [1] that prime numbers in short intervals are distributed normally, but with a variance that differs from classical heuristics.

In his 1998 Princeton PhD thesis, a part of which was published in *Annals of Mathematics* (2000), he showed that 7/8-ths of the quadratic L-functions have no zeros at the critical point $s = 1/2$ [2]. This was major progress towards a conjecture of S. Chowla, which is that $L(1/2, \chi) \neq 0$ for all quadratic L-functions. In collaboration with Brian Conrey, he proved [3] that a positive proportion of Dirichlet L-functions have no zeros on the real axis within the critical strip. The Generalized Riemann Hypothesis (GRH) is the statement that all non-trivial zeros of $L(s, \chi)$ lie on the critical line $\mathrm{Re}(s) = \sigma = 1/2$, but it is believed that $L(1/2, \chi) \neq 0$ for all primitive Dirichlet characters χ. In another paper in *Inventiones Mathematicae* [4], he and Ken Ono, assuming the GRH, confirmed a conjecture of Ramanujan regarding a ternary quadratic form.

Soundararajan continued to do influential work in analytic number theory, such as in the QUE conjecture with Roman Holowinsky (see work of Holowinsky below) and on the anatomy of the integers (with Andrew Granville).

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(c) 2006 winner Terence Tao

Terence Tao of UCLA had made far-reaching contributions to diverse areas of mathematics such as number theory, harmonic analysis, partial differential equations and ergodic theory. In making significant progress on long-standing problems, he has collaborated with a number of mathematicians.

A notable contribution was to the Kakeya problem in higher dimensions [1,2]. One aspect of the problem is to determine the fractal dimension of a set obtained by rotating a needle in n-dimensional space. In joint work [1] with Nets Katz, Isabella Laba and others, he improved all previously known estimates for the fractal dimension with ingenious combinatorial ideas. Another of Tao's seminal contributions was his joint work with Ben Green on long arithmetic progressions of prime numbers [3]. One of the deepest results in this area is a 1975 theorem of E. Szemeredi [4], which states that:

Any set of positive integers with upper positive density will contain arbitrarily long arithmetic progressions.

This was actually conjectured by Paul Erdös and Paul Turan in 1936 and first proved by Szemeredi. In 2001, Tim Gowers gave a new proof of a multi-dimensional version of Szemeredi's theorem. Szemeredi's theorem does not apply to the set of primes which is of zero density. By combining the ideas of Gowers along with tools from ergodic theory, Green and Tao proved the following sensational result:

The Green–Tao theorem. *There are arbitrarily long arithmetic progressions within the primes.*

This created a new field of mathematics called additive combinatorics.

Another fundamental contribution of Tao concerns the sum-product problem due to Erdös and Szemeredi: *Either the sum set or the product set of a set of N numbers must be large.* Tao was the first to recognize the significance of this problem in combinatorial number theory and harmonic analysis. In collaboration with Jean Bourgain and Nets Katz, Tao [5] made important generalizations and refinements of the Erdös–Szemeredi conjecture which have led to breakthroughs in harmonic analysis and number theory.

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(d) 2007 winner Ben Green

Ben Green had made outstanding contributions to several fundamental problems in number theory by himself and in collaboration with Terence Tao.

Green's PhD thesis at the University of Cambridge is a collection of several fundamental papers. In one of these papers [1] he solved the Cameron–Erdös conjecture, which is a bound on the number of sum-free sets of positive integers up to a given number N. Over the years several top mathematicians had worked on this problem, which was finally solved by Green.

A subset S of an Abelian group is said to be sum-free if the equation $x + y = z$ has no solutions with $x, y, z \in S$. Clearly, if S is sum-free, so would any subset of S.

The Cameron–Erdös conjecture (1990). *The number of sum-free subsets of* $\{1, 2, \ldots, N\}$ *is* $O(2^{N/2})$.

Motivation. The set of odd numbers in $\{1, 2, \ldots, N\}$ is sum-free. There are $\lceil N/2 \rceil$ odd numbers up to N. The number of subsets of these odd numbers is $2^{\lceil N/2 \rceil}$. Also, the set of integers in the interval $(N/2, N]$ is sum-free and this has $2^{\lceil N/2 \rceil}$ subsets as well. There are sum-free sets other than subsets of the two collections mentioned above. For example, when $N = 8$, the set $\{2, 3, 7, 8\}$ is sum-free. But the conjecture is that the number of sum-free subsets of the first N integers cannot exceed the order of magnitude of $2^{N/2}$.

Prior to collaborating with Tao, Green had established an important result [2] that any set of primes with relative upper positive density would contain infinitely many arithmetic progessions of length 3. This is a prime analogue of a theorem of Roth that any set of upper positive density contains infinitely many three-term arithmetic progressions. It was this paper of Green in *Annals of Mathematics* in 2005 that caught the attention of Tao and that led to their collaboration and to the Green–Tao theorem. Also of relevance is Green's important paper [3] on a Szemeredi-type regularity lemma. Green's 2005 paper [2] contained fundamental ideas that he and Tao could greatly develop and build on; they first extended Green's theorem to arithmetic progressions of length 4, and finally settled the long-standing conjecture [4] that there exist arbitrarily long arithmetic progressions among the primes. Subsequently, Green and Tao also collaborated in extending the Hardy–Ramanujan–Littlewood Circle Method by bringing in methods from ergodic theory [4,5]. The work of Green and Tao is having a major impact on analytic number theory.

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(e) 2008 winner Akshay Venkatesh

Akshay Venkatesh was making powerful contributions by himself and with a host of collaborators to diverse areas such as number theory, automorphic forms, representation theory, locally symmetric spaces and ergodic theory. His 2006 paper [1] with H. Helfgott provided the first non-trivial upper bound for the 3-torsion in class groups of quadratic fields. His joint work [2] with Jordan Ellenberg on representing integral quadratic forms by quadratic forms is spectacular and has its roots in the work of Ramanujan. Ellenberg and Venkatesh provided a striking application of Ratner's classification of measures invariant under unipotent flows to a central problem on quadratic forms, the scalar case of which was of interest to Ramanujan. The problem is that of representing a quadratic form in m variables by those in n variables; Ellenberg and Venkatesh obtained a local–global principle that provides bounds that are sharper than those obtained by earlier methods.

An important and difficult problem in number theory is to asymptotically count number fields according to their discriminant. This is a generalization of the classical problem of determining the relation between the number of rational or integral solutions of a polynomial equation in several variables and the coordinates of the solutions. The case up to degree 5 was solved by Manjul Bhargava. For large degrees, Ellenberg and Venkatesh provided the first major improvement over the bounds in earlier work of Wolfgang Schmidt and thus broke an impasse of 30 years.

Also of great importance is Venkatesh's joint work [3] with E. Lindenstrauss, which settled a famous conjecture of Peter Sarnak concerning locally symmetric spaces. One of Venkatesh's most impressive achievements was his individual work [4] on the sub-convexity of automorphic L-functions. The problem of sub-convex bounds at the centre of the critical strip for L-functions is very important. He provided a novel and more direct way of establishing sub-convexity in numerous cases that went beyond the foundational work of Hardy–Littlewood–Weyl and of other leading researchers.

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(f) 2009 winner Kathrin Bringmann

Bringmann was awarded the prize for fundamental work on modular forms and on mock theta functions by herself and with collaborators, most notably Ken Ono. Bringmann's PhD thesis at Heidelberg University in 2004 contained major results on two difficult problems: (i) the Ramanujan–Petersson conjecture for the coefficients of Siegel cusp forms whose weight is the dimension of the genus group plus one and (ii) generalizations of the work of Gross, Kohnen and Zagier on the existence of lifting maps between spaces of Jacobi forms and elliptic modular functions. This put her in a perfect position to study the relationship between mock theta functions and modular forms.

Mock theta functions were discovered by Ramanujan shortly before his death in 1920, and he communicated his findings in his last letter to Hardy in January that year. These are now considered among Ramanujan's deepest contributions.

Ramanujan's mock theta functions: these are objects which are like the classical theta functions in their shape, but are not modular forms, yet their coefficients can be calculated with a degree of precision comparable to what can be done for functions that can be expressed in terms of theta functions.

Ramanujan's definition of a mock theta function is not precise: 'It is a function $F(q)$ that satisfies the following conditions: (i) It can be written in Eulerian (q-hypergeometric) form and has singularities at infinitely many roots of unity. (ii) For each root of unity ζ, there is a theta function $f_\zeta(q)$ such that the difference $F(q) - f_\zeta(q)$ is bounded as $q \to \zeta$ radially. (iii) There is not a single theta function that satisfies (ii) for all roots of unity'.

Some examples of Ramanujan's mock theta functions

(A) The partition-generating function in 'transformed Eulerian form' (a term used by Ramanujan) is the following series having a product representation:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)}.$$

In providing a combinatorial explanation for two of Ramanujan's famous congruences for the partition function, Freeman Dyson (a young undergraduate at the University of Cambridge in 1944) considered a statistic he termed the *rank of a partition*, which is the largest part of a partition minus the number of parts. Dyson's generating function for the rank that generalizes the above series is

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-zq)(1-zq^2)\cdots(1-zq^n)\times(1-z^{-1}q)(1-z^{-1}q^2)\cdots(1-z^{-1}q^n)}.$$

Taking $z=1$ in Dyson's rank-generating function yields the transformed Eulerian series, whereas the choice $z=-1$ yields

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

which is a mock theta function of Ramanujan that we may write as a power series

$$\sum \alpha(n) q^n.$$

(B) The first Rogers–Ramanujan identity is

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-1})(1-q^{5m-4})}$$
$$= \frac{\prod_{m=1}^{\infty}(1-q^{5m})(1-q^{5m-2})(1-q^{5m-3})}{\prod_{m=1}^{\infty}(1-q^m)} = \frac{\sum_{-\infty}^{\infty}(-1)^n q^{(5n^2-n)/2}}{\sum_{-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}},$$

by Jacobi's Triple Product Identity for theta functions and Euler's Pentagonal Numbers Theorem. If we slightly modify the denominator terms in the series on the left, we get an example of a fifth-order mock theta function of Ramanujan, namely

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

No one knew the exact relationship between theta functions and mock theta functions, and indeed determining this relationship was one of the tantalizing puzzles of mathematics.

The first breakthrough in understanding mock theta functions was made by Sander Zwegers in 2003 in a thesis [1] written under the direction of Don Zagier in Bonn but submitted to Utrecht University. Zwegers used certain identities of George Andrews to rewrite mock theta functions in terms of Lambert series and indefinite theta series, and showed how mock theta functions fit into the theory of real analytic modular forms. From here Bringmann and Ono, and others in Ono's group, obtained far-reaching results, where the connections with modular forms are made explicit, questions concerning asymptotics and congruences are addressed and a comprehensive theory relating holomorphic cusp forms to modular forms are made.

The modern definition of a mock theta function is a function that is the holomorphic part f^+ of a weight 1/2 (or 3/2) harmonic Maass form $f = f^+ + f^-$, such that the non-holomorphic part f^- is a certain integral involving a weight 3/2 (or 1/2) unary theta function (a linear combination of theta functions). A harmonic Maass form is a function that transforms like a holomorphic modular form, but is annihilated by a certain hyperbolic Laplacian and satisfies a certain asymptotic condition near the cusps.

Bringmann and Ono [2], inspired by work of Zagier and Zwegers, wrote a fundamental paper, in which they show that all of Ramanujan's 22 (classical) mock theta functions are special cases of infinite families of weak Maass forms of weight 1/2. In yet another seminal paper, Bringmann and Ono [3] settled a 40-year-old conjecture of Andrews and Dragonette on the coefficients $\alpha(n)$ of the Ramanujan mock theta function alluded to above. More precisely, just as the partition function could be expressed in terms of the Rademacher convergent series by the use of the circle method, Andrews and Dragonette conjectured that there ought to exist an exact formula for the coefficients $\alpha(n)$. In the seme vein, yet another work of Bringmann–Ono deserves mention: in a 2007 paper [4], they define maps that lift holomorphic cusp forms of half-integral weight to harmonic weak Maass forms, and this theory includes weight 3/2 Maass forms which contain all of Ramanujan's mock theta functions. Finally, it should be mentioned that Bringmann also had major collaborations with several noted researchers in the area of mock theta functions such as Frank Garvan, Amanda Folsom, Karl Mahlburg, Jeremy Lovejoy and Sander Zwegers.

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(g) 2010 winner Wei Zhang

When Wei Zhang received the prize, he was Benjamin Pierce Instructor at Harvard University. He did his doctoral work under Prof. Shouwu Zhang at Columbia University, and had made seminal contributions by himself and in collaboration with others to a broad range of areas including number theory, automorphic forms, trace formulae, *L*-functions, representation theory and algebraic geometry.

In 1997 Steve Kudla constructed a family of cycles on Shimura varieties, and conjectured that their generating functions are actually Siegel modular forms. Richard Borcherds had proved the Kudla conjecture for cycles of codimension 1. In his PhD thesis, Wei Zhang established conditionally a generalization of Borcherds' result for higher dimensions and in that process essentially settled the Kudla conjecture. This thesis opened up major lines of research and led to significant collaboration with Xinyi Yuan and Shouwu Zhang. In the first of these joint papers [1], the results of Wei Zhang's thesis are generalized to totally real fiends. The three then established [2] an arithmetic analogue of a theorem of Waldspurger that connects integral periods to values of *L*-functions. This work goes well beyond all earlier work on formulae of the Gross–Zagier type.

In addition, Wei Zhang had made two other major contributions [3,4] on relative trace formulae and the Gross–Prasad conjecture, and on arithmetic fundamental lemmas. In these works, he made decisive progress on certain general conjectures related to the arithmetic intersection of Shimura varieties, and in that process successfully transposed major techniques due to Jacquet and Rallis into an arithmetic intersection theory setting. (The Gross–Prasad conjecture is a restricted branching problem in the representation theory of *p*-adic groups.)

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(h) 2011 winner Roman Holowinsky

Roman Holowinsky of the Ohio State University had made significant contributions to areas lying at the interface of analytic number theory and the theory of modular forms. Along with 2005 SASTRA Prize winner Soundararajan (Stanford), he solved an important case of the quantum unique ergodicity (QUE) conjecture. This was an impressive achievement.

1n 1991, Zeev Rudnick and Peter Sarnak formulated the QUE conjecture, which in general terms concerns the correspondence principle for quantizations of chaotic systems. One aspect of the problem is to understand how waves are influenced by the geometry of their enclosure. Rudnick and Sarnak conjectured that if the surface has negative curvature, then the high-frequency quantum wave functions are uniformly distributed within the domain. The modular domain in number theory is one of the most important examples, and for this domain Holowinsky and Soundararajan solved the QUE conjecture.

The manner in which the solution came about is amazing. Since 1991, many mathematicians had attacked this problem and made major advances. Luo and Sarnak reduced the problem to obtaining good estimates for certain shifted convolution sums. Elon Lindenstrauss had proved the QUE conjecture for Maass forms, but the problem for holomorphic domains remained open. By a study of Hecke eigenvalues and an ingenious application of the sieve, Holowinsky [1,2] obtained critical estimates for shifted convolution sums and this almost settled the QUE conjecture except in certain cases where the corresponding *L*-functions behave abnormally. Simultaneously, Soundararajan, who approached the problem from an entirely different direction, was able to confirm the conjecture in several cases, and noticed that the exceptional cases not fitting Holowinsky's approach were covered by his techniques. Thus by combining the approaches of Holowinsky and Soundararajan, the holomorphic QUE conjecture was fully resolved in the modular case [3]. Also of great interest is Holowinsky's joint work [4] with Blomer on bounding sup-norms of cusp forms of large level.

The QUE conjecture and its resolution is a fine example of great mathematical work inspired by a problem in physics. The QUE conjecture has connections with several important problems such as the GRH, Poincaré series, Maass forms, cusp forms and the Sato–Tate conjecture.

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(i) 2012 winner Zhiwei Yun

Zhiwei Yun has made fundamental contributions to several areas that lie at the interface of representation theory, algebraic geometry and number theory.

Yun's PhD thesis on global Springer theory at Princeton University opened up new vistas in the Langlands program. Springer theory is the study of Weyl group actions on the cohomology of certain subvarieties of the flag manifold called Springer fibres. Yun's global Springer theory deals with Hitchin fibres instead of Springer fibres (taking the lead from earlier work on Springer fibres by Gerard Lauman and Ngô Báo Châu), which he used to determine the actions of Weyl groups on cohomology [1,2]. This work led to a geometric and functorial understanding of the Langlands program.

Bao-Châu Ngô was awarded the 2010 Fields Medal for his proof of the Fundamental Lemma in the Langlands program. Yun made a breakthrough in the study of the Fundamental Lemma formulated by Jacquet and Rallis in their programme of proving the Gross–Prasad conjecture on relative trace formulae [2,3]. Yun's understanding of Hitchin fibrations enabled him to reduce the Jacquet–Rallis fundamental lemma to a cohomological property of Hitchin fibrations.

Yun has collaborated with Ngô and Jochen Heinloth on a seminal paper on Kloosterman sheaves for reductive groups [4]. Another work of Yun on the uniform construction of motives with exceptional Galois groups is considered to be fundamental. A construction like Yun's was sought by Fields Medallists Serre and Grothendieck for 40 years! Finally, Yun's work on the inverse Galois problem [5] used tools for the first time from the theory of automorphic forms, which are generalizations of modular forms that Ramanujan was a master of.

Since 2012 was the 125th Birth Anniversary of Ramanujan, Yun was awarded the SASTRA Prize in India's capital New Delhi at a conference organized by the National Board for Higher Mathematics (of India) and co-sponsored by SASTRA University and Delhi University. That was the only year when this prize was not awarded in Kumbakonam.

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(j) 2013 winner Peter Scholze

Peter Scholze has made revolutionary contributions to areas at the interface of arithmetic algebraic geometry and the theory of automorphic forms. Already in his Masters thesis at Bonn, he gave a new proof of the local Langlands conjecture for general linear groups [1]. There were two previous approaches to this problem, one by Langlands and Kottwitz and another by Harris and Taylor. Scholze's new approach was striking in its efficiency and simplicity—it was based on a novel approach to calculate the zeta function of certain Shimura varieties. This work, completed in 2010, appeared in two papers [2,3] in 2013. He then generalized this (partly in collaboration with Sug Woo Shin) to determine ℓ-adic Galois representations defined by a class of Shimura varieties [4,5].

His PhD thesis was a more marvellous breakthrough. There he developed a new p-adic machine called *perfectoid spaces* and used it to prove a significant part of the weighted monodromy conjecture due to Deligne, thereby breaking an impasse of more than 30 years. This work [6] appeared in 2012. He then extended his theory of perfectoid spaces to develop a Hodge theory [2] for rigid analytic spaces over p-adic ground fields, generalizing a theory due to Faltings for algebraic varieties. As a consequence, he answered a question on spectral sequences that John Tate had raised four decades earlier. A key igredient in Scholze's approach is his construction of a *pro-etale site*; this has led to a new foundation for etale cohomology that he investigated with Bhargav Bhatt.

Yet another seminal work of Scholze is his collaboration with Jared Weinstein [7] extending earlier results of Rapoport–Zink on moduli spaces of p-divisible groups; they show that, at an infinite level, these carry the structure of a perfectoid space. One significant consequence of this is that it yields a simple description of p-divisible groups over the ring of integers of an algebraically closed extension of the p-adic rationals which is analogous to Riemann's description of (what are now called) Abelian varieties over the complex numbers.

A further major achievement of Scholze [8] was to show the existence of Galois representations associated with the mod p cohomology of locally symmetric spaces for linear groups over a totally real or *CM* field. This has had surprising implications for the Betti cohomology of locally symmetric spaces.

One may say that Scholze studies mysterious relations between modular forms and their generalizations to other mathematical objects. One central question is Ramanujan's conjecture bounding the size of the Fourier coefficients of modular forms. The weight monodromy conjecture that Scholze attacked in his PhD thesis can be regarded as an arithmetic analogue of it.

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(k) 2014 winner James Maynard

James Maynard of the University of Oxford had made spectacular contributions to some of the most famous problems on prime numbers. The theory of primes is an area where questions which are simple to state can be very difficult to answer. A supreme example is the 'prime twins conjecture', which states that there are infinitely many primes that differ by 2. This is still unsolved. Maynard obtained the strongest results at the time on the prime twins conjecture by showing that the gap between consecutive primes does not exceed 600 infinitely often. Not only did this improve upon the earlier path-breaking work of Goldston–Pintz–Yildirim, and of Zhang, but he achieved this with ingenious methods which were simpler [1]. His success on the 'small gaps problem' for primes was built upon ideas in his PhD thesis pertaining to the Brun–Titchmarch inequality, which concerns the distribution of primes in arithmetic progressions [2]; the problem becomes difficult when the modulus of the progression gets large, and the results are weaker than the conjectured size. Earlier researchers had relied on certain unproved hypotheses concerning the Siegel zeros of L-functions to treat large moduli. Maynard showed in his thesis, without appeal to any hypothesis, that the number of primes in arithmetic progressions is bounded by the suspected heuristic size for arithmetic progressions whose moduli do not exceed the positive eighth root of the largest member of the progression.

A generalization of the prime twins conjecture is the prime k-tuples conjecture, which states that an admissible collection of k linear functions will simultaneously take k prime values infinitely often. In the last 100 years, several partial results towards the k-tuples conjecture were obtained by replacing prime values with 'almost primes', namely numbers with a bounded number of prime factors, or by bounding the total number of prime factors in the product of these linear functions.

Another major achievement of Maynard in his doctoral thesis and in his post-doctoral work was to significantly improve on the work of earlier researchers on k-tuples of almost primes [3]. For instance, one of his results was that the product three admissible linear functions will have at most seven prime factors infinitely often [4]. Maynard's results and methods have led to a resurgence of activity in prime number theory.

One month before receiving the SASTRA Prize, Maynard announced the solution to the famous $10 000 problem of Erdös on large gaps between primes [5]. This was simultaneously announced by Kevin Ford, Ben Green, Sergeo Konyagin and Terence Tao, but Maynard's method was different and simpler. The prime number theorem implies that the 'average gap' g_n between the nth and $n+1$st prime is asymptotically $\log n$ in size. In 1938, Robert Rankin established that there exists a positive constant c such that, infinitely often,

$$g_n > c.\frac{\log n \log\log n \log\log\log\log n}{(\log\log\log n)^2},$$

which remained for many years the best result on the large gap problem for primes. The great Hungarian mathematician Paul Erdös, known for stating problems and offering prize money for their solutions, asked whether the constant c can be made arbitrarily large, and offered $10 000 for the resolution of this question either in the affirmative or in the negative. Maynard (and independently Ford–Green–Konyagin–Tao) showed that the constant c can be made arbitrarily large by replacing $(\log\log\log n)^2$ in the denominator with $\log\log\log n$.

Another fundamental work of Maynard concerns limit points of the sequence $g_n/(\log n)$. Results on the small gaps problem show that 0 is a limit point, while the theorems on the large gap problem imply that ∞ is a limit point. No other limit point is known. In collaboration with William Banks and Tristan Freiburg, Maynard [6] has shown that if any 50 random positive real numbers are given, then at least one of the 50 consecutive gaps between them and 0 will occur as a limit point. Both Maynard's results and methods have had far-reaching implications.

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(l) 2015 winner Jacob Tsimerman

Jacob Tsimerman has made deep contributions to diverse parts of number theory, most notably to the André–Oort conjecture. He has mastery over two very different areas of mathematics—analytic number theory and algebraic geometry—and has made significant progress on a number of fundamental questions lying at the interface of the two subjects.

The André–Oort conjecture states that special subsets of Shimura varieties that are obtained as Zariski closures of special points are finite unions of Shimura varieties. Shimura varieties are special algebraic varieties (such as modulus spaces of Abelian varieties) which arise as quotients of suitable complex domains by arithmetic groups. Thus they lie at the heart of arithmetic geometry and automorphic forms. Yves André initially stated this conjecture for one-dimensional sub-varieties, and subsequently Franz Oort proposed that it should hold more generally. By assuming the GRH, the conjecture was proved in 2006 by Klinger, Ullmo and Yafaev, but as of 2008 only the very simplest cases had been proved unconditionally. One of the techniques to attack this conjecture is to obtain suitable bounds for certain Galois orbits of special points. A major achievement of Tsimerman in his Princeton PhD thesis of 2011 was to obtain certain unconditional bounds up to dimension 6 [1].

Another very important result in his thesis was to answer in the affirmative a question due to Nick Katz and Oort whether there exists an Abelian variety over the set of all algebraic numbers which is not isogenous to the Jacobian of a stable algebraic curve over the algebraic numbers [2]. Previously, Ching-Li Chai and Frans Oort had answered the question assuming the André-Oort conjecture, but Tsimerman was able to do so unconditionally.

About a decade ago, Jonathan Pila introduced new methods to attack the André–Oort conjecture. In 2009, Tsimerman joined forces with Pila and established several deep results [3,4], one of which was a functional transcendence statement known as 'Ax-Lindemann' for Abelian varieties of all dimensions. (Ax-Lindemann is one of the tools to attack the André–Oort conjecture.)

Just before receiving the SASTRA Prize, Tsimerman gave a proof of the André-Oort conjecture for the moduli spaces of principally polarized Abelian varieties of any dimension—a result that was sought for a long time.

Even as a graduate student at Princeton, Tsimerman [5] collaborated with Manjul Bhargava and Arul Shankar to determine the second term in the asymptotic formula for the number of cubic fields with a bounded discriminant. Especially relating to Ramanujan's mathematics, we note his

2014 paper [6] joint with Ali Altug on the metaplective Ramanujan conjecture over function fields, which has applications to quadratic forms.

Tsimerman says: 'My work on the André–Oort conjecture crucially relied on a deep expression for the Faltings height of an abelian variety due to Clozel. The formulation of this expression, as well as its proof due to Yuan and Zhang, rely crucially on theta functions, a subject pioneered and greatly developed by Ramanujan. Indeed, Ramanujan had a keen understanding that analytic bounds could be obtained for fundamental arithmetic quantities by expressing them in a modular framework, as is done so vividly by his work with Hardy on the partition function, and it is partially this insight that led to his work on theta functions in the first place. In the modern context, a generalization known as the theta correspondence plays the role of the theta function, and allows one to trade in one complicated expression involving heights for another involving L-functions, which can be more easily bounded.'

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(m) 2016 winners Kaisa Matomäki and Maksym Radziwill

Matomäki and Radziwill were awarded the prize for their revolutionary collaborative work on multiplicative functions in short intervals. This shocked the mathematical community by going well beyond what could be proved previously even assuming the Riemann hypothesis, and opened the door for a series of breakthroughs on some notoriously difficult questions such as the Erdös discrepancy problem and Chowla's conjecture, previously believed to be well beyond reach.

Their collaboration began in 2014 with a paper [1] on the sign changes of Hecke eigenvalues. This led to their joint work [2] on multiplicative functions in short intervals, and especially that of the Liouville lambda function $\lambda(n)$, which takes value 1 when an integer n has an even number of prime factors (counted with multiplicity) and value -1 when n has an odd number of prime factors. The statement that $\lambda(n)$ takes values 1 and -1 with asymptotically equal frequency is equivalent to the prime number theorem; more refined statements on the relative error in this equal frequency are related to the Riemann hypothesis.

Such equi-distribution results were known also for short intervals $[x, x+h]$, with h being a fractional power (less than 1) of x. One would expect that such equal frequency would hold when h is of size x^{ε}, where ε is arbitrarily small and positive, but even with the Riemann hypothesis the best that is known is that powers larger than 1/2 will work. Instead of asking for equal frequency in every short interval of length h, if we require that equal frequency should hold for 'almost all' short intervals of length h, then one can reduce the size of h considerably: it was shown that h can be made essentially of size of the 1/6th power of x, and, assuming the Riemann hypothesis, Gao showed that h can be taken as small as a power of $\log x$. Matomäki and Radziwill shocked the world by showing unconditionally that equal frequency holds almost always as long as h tends to infinity with x. The ideas introduced by Matomäki and Radziwill in achieving this are expected to transform the subject in a major way. Since their paper in the *Annals of Mathematics* deals more generally with multiplicative functions, there are many other significant implications, such as the distribution of smooth numbers in short intervals.

Smooth numbers are those which are devoid of large prime factors and they figure prominently in the study of factorization algorithms. The problem of the distribution of smooth numbers in short intervals is important and challenging. A corollary of their main theorem on multiplicative functions in short intervals is that, given any small positive quantity ϵ, there exists a constant $C(\epsilon)$ depending only on ϵ such that every short interval $[x, x+h]$ of length $h = C(\epsilon)\sqrt{x}$ will contain a smooth number whose prime factors are all less than x^{ε}. Previously such a result on smooth numbers was only known conditionally, that is, assuming the Riemann hypothesis, and it is remarkable that Matomäki and Radziwill established this unconditionally.

Sarvadaman Chowla had conjectured that if any k collection of values 1 and -1 are given in any order, then the lambda function will take that sequence of values at k consecutive integers with asymptotic frequency $1/2^k$. This conjecture is as yet unsolved. But very recently Matomäki, Radziwill and Terence Tao have proved that, when $k = 3$, each of the eight sign choices occurs with positive proportion (probability) [3,4].

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(n) 2017 winner Maryna Viazovska

Maryna Viazovska of the Swiss Federal Institute of Technology, Lausanne, was awarded the 2017 SASTRA Prize for her stunning solution in dimension 8 of the sphere-packing problem, and for her equally impressive joint work with Henry Cohn, Abhinav Kumar, Stephen D. Miller and Danylo Radchenko in resolving the sphere-packing problem in dimension 24, by building upon her fundamental ideas in dimension 8.

Viazovska has made deep contributions to several fundamental problems in number theory. In her PhD thesis of 2013 at the Max Planck Institute for Mathematics at the University of Bonn, she resolved the famous Gross–Zagier conjecture in a substantial number of cases, including the important case pertaining to higher Green's functions that had been open for 30 years. Her thesis work was a tour de force, making use of a variety of tools such as Borcherds lifts, Kudla's programme on the arithmeticity of theta correspondences and clever technical calculations.

Prior to her PhD, she had a number of impressive publications in collaboration with several active researchers. Particularly significant is her joint work [1] with Andriy Bondarenko and Danylo Radchenko, which resolved a long-standing conjecture of Korevaar and Meyers on spherical designs, by giving an optimal upper bound for the minimal number of points in a spherical design. It is with this background that she started thinking about the sphere-packing problem while in Bonn.

The sphere-packing problem has a long and illustrious history. Johannes Kepler asked for the optimal way to assemble cannon balls (of uniform radius) and conjectured a configuration, but could not prove it. This is the sphere-packing problem in three dimensions, and can be generalized to arbitrary dimensions. The sphere-packing problem arises naturally not just in geometry and physics but also in information theory, where sphere packings are error-correcting codes for a continuous communication channel.

The sphere-packing problem in dimension 3, namely the Kepler conjecture, was resolved by Thomas Hales [2] in 1998 by combining ingenious geometric optimization arguments with machine calculations. The sphere-packing problem in higher dimensions remained open.

In dimension 8, there is E_8, an exceptional Lie group with a root lattice of rank 8, and in dimension 24 there is the Leech lattice; both have remarkable structures. This gave some hope that the sphere-packing problem could be solved in dimensions 8 and 24. Noam Elkies and Henry Cohn made significant progress by using the Poisson summation formula and linear programming bounds for the sphere-packing density. They conjectured the existence of certain magic auxiliary functions in dimensions 8 and 24, which, if determined, would resolve the conjecture in these dimensions. Viazovska produced these functions in dimension 8 by ingenious use of modular forms. Her attack was viewed as audacious, but, when she succeeded, the mathematical world applauded in disbelief because her proof [3] is remarkably simple.

Paul Erdös had jocularly remarked that God has a book of the most beautiful proofs of the most significant results. Viazovska's proof is considered to belong to this book of God!

Once Viazovska had succeeded in dimension 8, the immediate question was whether her methods could be extended to dimension 24. Within a span of one week, working at a furious pace, by extending the ideas in dimension 8, the sphere-packing problem in dimension 24 was resolved by Cohn, Kumar, Miller, Radchenko and Viazovska [4].

Viazovska's work makes crucial use of the theory of modular forms, which were a favourite of Ramanujan's. Viazovska's modular form techniques are by no means limited to the sphere-packing problem. She has discovered something profound that will play a broader role in discrete geometry, analytic number theory and harmonic analysis.

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(o) 2018 winners Yifeng Liu and Jack Thorne

The 2018 SASTRA Ramanujan Prize was jointly awarded to Yifeng Liu (Yale University) and Jack Thorne (University of Cambridge).

Yifeng Liu

He has made deep contributions to arithmetic geometry and number theory spanning a wide spectrum of topics such as arithmetic theta lifts and derivatives of *L*-functions, the Gan–Gross–Prasad conjecture and its arithmetic counterpart, the Beilinson–Bloch–Kato conjecture, the geometric Langlands program, the *p*-adic Waldspurger theorem and the study of étale cohomology on Artin stacks.

His research began with his PhD thesis at Columbia University, written under the direction of Professor Shouwu Zhang. He went well beyond the work of Kudla, Rapoport and Yang on *L*-series connected with the minimal level for Shimura curves over the rationals [1,2]. Subsequently, in three papers [3–5] pertaining to Bessel and Fourier–Jacobi models that appeared in the *Journal of Functional Analysis* in 2013 (co-authored with Binyong Sun), *Manuscripta Mathematica* in 2014 and *Journal für die reine und angewandte Mathematik* in 2016, he made real progress on the famous Gan–Gross–Prasad conjectures in the representation theory of classical groups.

In collaboration with Shouwu Zhang and Wei Zhang, he studied the *p*-adic logarithm of Heegner points and proved [6] *p*-adic versions of theorems of Waldspurger and of Gross–Zagier; this is a vast generalization of previous important work of Bertolini, Darmon and Prasanna.

In the 1980s, Gross–Zagier and Kolyvagin proved some amazing theorems that implied the celebrated Birch–Swinnerton–Dyer conjecture for elliptic curves when the analytic rank is either 0 or 1. Liu has successfully extended Kolyvagin-type results to higher ranks in the more general framework of the Beilinson–Bloch–Kato conjecture [7,8]. Liu has also made inroads into non-Archimedean geometry; he established [9] a non-Archimedean analogue of the famous Calabi conjecture for Abelian varieties over certain p-adic fields.

Liu has said: 'A large part of my work is about automorphic forms, which are generalizations of modular forms to higher rank groups. Modular forms, or rather in the form of q-series, are a major part of Ramanujan's work. For example, the famous Ramanujan conjecture is a deep speculation on the size of coefficients of modular forms. My works on the Gan–Gross–Prasad conjecture study period integrals in terms of L-values, which are important analytic invariants attached to q-series. In particular, lots of study concerns a large class of automorphic forms that satisfy analogues of the Ramanujan conjecture, known as tempered automorphic forms. My works on arithmetic theta lifting and the Beilinson–Bloch–Kato conjecture focus on the relation between L-values of automorphic forms with arithmetic objects, like algebraic cycles on varieties over number fields, and Selmer groups.'

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Jack Thorne

Jack Thorne has made fundamental contributions to number theory, representation theory and algebraic geometry. He works in two rather different areas: modularity of Galois representations and arithmetic invariant theory. His 2012 PhD thesis 'The arithmetic of simple singularities' at Harvard University was jointly supervised by Professors Richard Taylor and Benedict Gross, two of the dominant figures in contemporary number theory. One outcome of the thesis was his work [1] on arithmetic invariant theory which, among other things, leads to new results about the sizes of Selmer groups for Abelian varieties of small dimension, and bounds for the number of rational and integral points on various types of algebraic curves of genus greater than 1. Arithmetic invariant theory concerns the application of geometric invariant theory to number theory. The most significant first applications to elliptic curves were carried out by Manjul Bhargava and his collaborators, but Thorne's work attempts to go beyond this.

Regarding modularity of Galois representations, Thorne has been a central force in eliminating restrictions on the Taylor–Wiles method. What is called the Taylor–Wiles method is a collection of techniques for proving that an ℓ-adic representation of the Galois group over a global field is automorphic.

Some of Thorne's most striking results have appeared in three papers [2–4], with Laurent Clozel. The main input in Thorne's work with Clozel is a new method due to Thorne for eliminating the most stubborn restriction in the Taylor–Wiles method, namely the assumption that the residual (mod ℓ) Galois representation under consideration is irreducible. An aspect of Thorne's ideas that was crucial in his joint work with Clozel was his startling discovery of a automorphy lifting theorem [5].

Thorne's 2015 work [6] with Khare on potential automorphy and the Leopoldt conjecture introduced a 'trick' that plays a key role in an ongoing 10-author collaboration including Thorne that will establish a potential version of the Shimura–Taniyama conjecture for elliptic curves over imaginary quadratic fields. Thorne's recent paper [7] establishing that all elliptic curves over $\mathcal{Q}_\infty$ are modular is viewed as another major breakthrough.

Thorne has said: 'The aspect of my work most closely related to that of Ramanujan is probably in the ten author collaboration you refer to. In this work, we prove the generalized Ramanujan conjecture (GRC) for Bianchi modular forms.

Ramanujan's original conjecture concerned the size of the Fourier coefficients of the discriminant modular form (the unique cuspidal modular form of level 1 and weight 12 with leading term q). Deligne saw how to translate it into algebraic geometry, where it would follow from the Weil conjectures, which he later proved. The GRC concerns an analogous statement for all cuspidal automorphic forms on general linear groups; Bianchi modular forms are the first such examples of such forms which do not have a direct link to algebraic geometry of the type exploited by Deligne, and our work is (I believe) the first to establish new cases of the GRC without reducing them to the Weil conjectures.'

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3. Shaping the development of mathematics

The Fields Medals, with an age limit of 40 for the winners, were instituted with two lofty goals in mind: (i) to recognize pioneering work by brilliant young mathematicians and (ii) to encourage these young researchers to continue to influence the growth of mathematics. The Fields Medallists have lived up to these great expectations. The SASTRA Ramanujan Prize has a more stringent age limit of 32, and so has recognized brilliant mathematicians even earlier in their careers.

It is no exaggeration to say that the winners of the SASTRA Ramanujan Prizes have also shaped the development of mainstream mathematics and will continue to do so in the years ahead. The fact that the SASTRA laureates have subsequently won major prizes with a hallowed tradition is a testimony to this; for instance, Fields Medal Winners Manjul Bhargava (2014), Peter Scholze (2018) and Akshay Venkatesh (2018) were all earlier SASTRA Prize Winners.

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