MY ASSOCIATION AND COLLABORATION WITH GEORGE ANDREWS*
Still going strong at 80

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0. Abstract

This is recollection of my association with George Andrews from 1981, and a report of my joint work with him in the theory of partitions and q-series relating to the Göllnitz and Capparelli theorems starting from 1990.

1. Introduction

George Andrews is the undisputed leader on partitions and the work of Ramanujan combined. After Hardy and Ramanujan, he, more than anyone else in the modern era, is responsible for making the theory of partitions a central area of research. His book on partitions [15] published first in 1976 as Volume 2 of the Encyclopedia of Mathematics (John Wiley), is a bible in the field, and his NSF-CBMS Lectures [16] of 1984–85 highlight the fundamental connections between partitions and Ramanujan's work with many allied fields. We definitely owe to him our present understanding of many of the deep identities in Ramanujan's Lost Notebook. I had the good fortune to collaborate with him and also interact with him very closely both at Penn State University (his home turf) where I visited often, and at the University of Florida, where he has spent the Spring term every year since 2005. I also have had the pleasure of hosting him in India several times. Thus I have come to know him really well as a mathematician, colleague, and friend. Here I will first share with you (in Section 2) my observations of him as a man and mathematician. I will then describe (in Section 3) some aspects of our joint work that will highlight his vast knowledge and brilliance. In Section 2, I will describe events chronologically rather than thematically. In Section 3, I will discuss my joint work with him on the Capparelli and the Göllnitz theorems.

2. Personal recollections

First visit to India: Even though Andrews has been studying Ramanujan's work since the sixties and had been "introduced to India" through the writings of, and on, Ramanujan, his first visit to India was only in Fall 1981. That academic year, I was visiting the Institute for Advanced Study in Princeton, and he contacted me saying that he was planning a visit to India, and to Madras in particular. My father, the late Professor Alladi Ramakrishnan, was Director of MATSCIENCE, The Institute of

*This is a mildly modified version of my paper "My association and collaboration with George Andrews - Torchbearer of Ramanujan and partitions" that appeared in *Lattice Paths Combinatorics and Applications* (G. E. Andrews, C. Krattenthaler, and A. Krinik, Eds.), Proc. 2015 Conf. in Pomona, *Developments in Mathematics*, **58** Springer, New York (2019). The permission to reproduce is gratefully acknowledged.

Mathematical Sciences, that he had founded in 1962, and so I put him in touch with my father who hosted him in Madras and helped arrange a meeting for Andrews with Mrs. Janaki Ammal Ramanujan. Upon return from India, Andrews called me from Penn State, told me that it was an immensely enjoyable and fruitful visit, and that he appreciated my father's help and hospitality. To reciprocate, Andrews invited me to a Colloquium at Penn State where he was Department Chair at that time. Andrews is always a gracious host, but in his capacity as Chair, he rolled out the red carpet for me! He hosted a party for me at his house during that visit and that is how our close friendship began.

I was working at that time in analytic number theory but I wanted to learn partitions and q-series, and that aspect of the work of Ramanujan. So after I returned to India from Princeton, I wrote to Andrews and asked him for his papers. Promptly I received two large packages containing more than 100 of his reprints. So I started studying them along with his Encyclopedia, and gave a series of lectures at MATSCIENCE in Madras, the notes of which I still use today. Even after this course of lectures, I was unsure whether to venture into partitions and q-series. The infinite series formulae were beautiful, but daunting. The decision to change my field of research to the theory of partitions and q-series came during the Ramanujan Centennial in Madras in December 1987.

The Ramanujan Centennial: The Ramanujan Centennial was an occasion when mathematicians from around the world gathered in India to pay homage to the Indian genius. Among the mathematical luminaries at the conference, there was a lot of attention on Andrews, Richard Askey and Bruce Berndt — jokingly referred to in the USA as the "Gang of Three" in the world of Ramanujan. I prefer to refer to them as the "Great Trinity" of the Ramanujan world, like Brahma, Vishnu, and Shiva, the three premier Hindu gods! The Great Trinity along with Nobel Laureate Astrophysicist Subrahmanyam Chandrasekhar and Fields Medalist Atle Selberg, were the stars of the Ramanujan Centennial. But Andrews occupied a special place in this elite group, because the Lost Notebook that he unearthed at the Wren Library in Cambridge University, was released in published form [21] at a grand public function in Madras on December 22, 1987, Ramanujan's 100-th birthday, by India's Prime Minister Rajiv Gandhi, who handed one copy to Janaki Ammal and another to Andrews. That definitely was a high point in the academic life of Andrews. Andrews has written a marvelous Preface to that book published by Narosa, which at that time was part of Springer, India.

December 1987 was a politically tense time in Madras because the Chief Minister of Madras, M. G. Ramachandran — MGR as he was affectionately known — a former cine hero to the millions, was terminally ill. There were several conferences in India around Ramanujan's 100-th birthday, and Andrews was a speaker in every one of them. He therefore arrived in Madras about a week before the 100-th birthday of Ramanujan and spent the first night at my house before traveling by road to a conference at Annamalai University, south of Madras. I told him that he should be very careful traveling by road in such a tense time, but he held my hand and said: "Krishna, do not worry. I am on a pilgrimage here to pay homage to Ramanujan. I will not let anything perturb me." As it turned out, one day as he, Askey, and Berndt were traveling by car on their way back to Madras, the car was suddenly encircled by a crowd of excited political activists. The car was stopped. Askey and Berndt were very nervous. But Andrews, cool as a cucumber, rolled down the window, and threw a load of cash into the air! The crowd cheered and let the car through because the foreigners had supported their cause. Andrews acted like James Bond, with tremendous presence of mind! Anyway, everyone made it safely to Madras for a one day conference I had arranged on December 21, and for the the December 22 function presided by Prime Minister Rajiv Gandhi.

The talks that Andrews gave at various conferences, including the one that I organized at Anna University on December 21, one day before the 100-th birthday of Ramanujan, were all for expert audiences. Since Andrews is a charismatic speaker, I wanted him to give a lecture to a general audience. So my father and I arranged a talk by him at our home on December 23, under the auspices of the Alladi Foundation that my father started in 1983 in memory of my grandfather Sir Alladi Krishnaswami Iyer, one of the most eminent lawyers of India. We invited the Consul General of the USA to preside over the lecture which was attended by prominent citizens of Madras in various walks of life — lawyers, judges, aristocrats, businessmen, college teachers and students. Andrews charmed them all with his inimitable description of the story of the discovery of Ramanujan's Lost Notebook. But something sensational happened that night after Andrews' lecture: Following the talk, many of us assembled at the Taj Coromandel Hotel for a dinner in honor of the conference delegates hosted by Mr. N. Ram, Editor of *The Hindu*, India's National Newspaper, based in Madras. (Ram's connection with Andrews was that in 1976, shortly after the Lost Notebook was discovered, he published a full page interview with Andrews in The Hindu.) After dinner, while we chatting over cocktails and dessert, the news came in whispers that MGR had passed away, and so the city would come to a standstill by daybreak once the general public would hear this news. So under the cover of darkness, we were asked to quietly make our way back to our hotels. And yes, as predicted, there was a complete shutdown and the Ramanujan Centenary Conference did not take place on December 24; instead all talks were squeezed into the next two days. Fortunately, Andrews had spoken at the conference on December 23. The Goddess of Namakkal had made sure that the Ramanujan Centenary celebration on December 22, and the talks the next day by the Great Trinity, would not be affected by such a tragedy!

The Frontiers of Science Lecture in Florida: At the University of Florida in Gainesville, there was a public lecture series called *Frontiers of Science*. This was organized by the physics department, and students received 1 (hour) course credit for attending these lectures. Many world famous scientists spoke in this lecture series such as group theorist John Conway, and Johansson, the discoverer of the "Lucy" skeleton. So after my return from the Ramanujan Centennial, I suggested to the organizers to invite George Andrews. I never heard back from them and so I felt they were not interested. Quite surprisingly, three years later, in Fall 1990, they contacted me and expressed interest in Andrews delivering a Frontiers of Science Lecture. So Andrews gave such a talk in November 1990, and held the 1000 or more members of the audience in the University Auditorium in rapt attention as he described the story of the discovery of the Lost Notebook. That was his first visit to Florida, but in that visit, our collaboration began in a remarkable way. I will now relate this fascinating story that will reveal the genius of this man.

In early 1989, I got a phone call from Basil Gordon, one of my former teachers at UCLA where I did my PhD work. Gordon said that he would be on a fully paid sabbatical in 1989–90, and that he would like to spend the Fall of 1989 in Florida. After the Ramanujan Centennial, I attempted some research on partitions and q-series, but the visit of Gordon provided me a real opportunity because Gordon was a dominant force in this domain; in the 1960s he had obtained a far-reaching generalization of the Rogers–Ramanujan identities to all odd moduli. Gordon and I first obtained a significant generalization of Schur's famous 1926 partition theorem by a new technique which we called the *method of weighted words*. We then extended this method to obtain a generalization and refinement of a deep 1967 partition theorem of Göllnitz. We cast this generalization in the form of a remarkable three parameter q-hypergeometric *key identity* which we were unable to prove. When Andrews arrived in Florida for the Frontiers of Science Lecture, I went to the airport to receive him. I did not waste any time and showed him the identity right there. He said it was fascinating. During his three day stay in Gainesville, he thought of nothing else. He focused solely on the identity. In the visitors office that he occupied in our department, I saw him working on the identity, every day, and every hour. On the last day, on the way to the airport, he handed me an eight page proof of this key identity by q-hypergeometric techniques that only he could wield with such power. That is how my first paper with him (jointly also with Gordon) came about.

Sabbatical at Penn State, 1992–93: I was having my first sabbatical in 1992–93 and Andrews invited me to Penn State for that entire year. So I went to State College, Pennsylvania with my family. It was the most productive year of my academic life — I completed work on five papers of which two were in collaboration with Andrews. He and his wife Joy were gracious hosts. They showed us around State College and we got together as families for picnics. Most importantly, Andrews gave a year long graduate course on the theory of partitions that I attended. Although I was doing research in the theory of partitions, I never had a course on partitions and q-hypergeometric series as a student and so it was a treat for me to learn from the master. Dennis Eichhorn and Andrew Sills were also taking this course as graduate students.

The sabbatical year at Penn State gave me time to also write up work I had done previously. It was there that I finished writing my first joint paper with Andrews on the Göllnitz theorem. The story of my second joint paper with Andrews written at Penn State on the Capparelli conjecture is also equally remarkable, and demonstrates once again Andrews' power in the area of partitions and q-hypergeometric series, and so I will relate this now.

In the summer of 1992, the Rademacher Centenary Conference was held at Penn State. Andrews was a former student of Rademacher, and so he was the lead organizer of this conference. On the opening day of the conference, Jim Lepowsky gave a talk on how Lie algebras could be used to discover, and in some instances, prove, various Rogers–Ramanujan type partition identities. During the talk, he mentioned a pair of partition identities that his student Stefano Capparelli had discovered in the study of vertex operators of Lie algebras but was unable to prove. Even though Andrews was the main conference organizer, he went into hiding during the breaks to work on the Capparelli Conjecture. By the end of the conference, he had proved the conjecture; so on the last day, he changed the title of his talk and spoke about a proof of the

Capparelli conjecture! This story bears similarity to the way in which he proved the three parameter identity for the Göllnitz theorem that Gordon and I had found but could not prove.

I was not present at the Rademacher Centenary Conference since I was in India at that time, just two months before reaching Penn State for my sabbatical. But Basil Gordon was at that conference and he told me this story. Actually, during Lepowsky's lecture, Gordon realized that our method of weighted words would apply to the Capparelli partition theorems and he expressed this view to me in a telephone call soon after I arrived at Penn State. So during my sabbatical, I worked out the details of this approach to obtain a two parameter refinement of the Capparelli theorems, and in that process got a combinatorial proof as well. This led to my second joint paper with Andrews, with Gordon also as a co-author.

Honorary doctorate at UF in 2002: In view of his fundamental research and his contributions to the profession, Andrews is the recipient of numerous honors. He has received honorary doctorates from the University of Illinois and the University of Parma. In 2002, he was awarded an Honorary Doctorate by the University of Florida. I was Department Chair at that time, and it was then that we formalized the arrangement to have him as a Distinguished Visiting Professor, so that he would spend the entire Spring Term each year at the University of Florida. Some of his most important recent works have had a Florida origin, such as his work on Durfee symbols, and on the function $spt(n)$.

Visit to SASTRA University, 2003: In 2003, the recently formed SASTRA University, purchased Ramanujan's home in Kumbakonam, renovated it, and decided to maintain it as a museum. This was a major event in the preservation of Ramanujan's legacy for posterity. To mark the occasion, SASTRA decided to have an International Conference at their newly constructed Srinivasa Ramanujan Centre in Kumbakonam to coincide with Ramanujan's birthday, December 22. I was invited to organize the technical session and given funds to bring a team of mathematicians to Kumbakonam. SASTRA was a new entry in the Ramanujan world, but this conference seemed to me interesting and promising. But how to make a success of this? So I called Andrews and told him that something exciting is happening in Ramanujan's hometown, and I would like him to give the opening lecture at this conference. He readily agreed. Once he accepted, I called other mathematicians and told them that Andrews will be there. So they too accepted the invitation to the First SASTRA Conference. That shows Andrews' drawing power! That conference was inaugurated by India's President Abdul Kalam who also declared open Ramanujan's home as a museum and national treasure.

Ramanujan 125, Honorary Doctorate at SASTRA: Many things developed after that 2003 SASTRA conference — the conferences at SASTRA became an annual event that I help organize, and in 2005 the SASTRA Ramanujan Prize was launched. SASTRA invited me to be Chair of the Prize Committee. I felt that Andrews' input would be crucial for the success of the prize. So I invited him to be on the Prize Committee during the first year, and he readily agreed. I then informed others about the prize and that Andrews was on the Prize Committee, and they too agreed enthusiastically. The prize as you know has become one of the most prestigious in the world, and

I am grateful to Andrews for agreeing to serve on the Prize Committee during the first year.

In view of the annual conferences and the prize, SASTRA had become a major force in the world of Ramanujan by the time Ramanujan's 125-th Anniversary was celebrated in December 2012. So I suggested to the Vice-Chancellor of SASTRA, that the three greatest figures in the world of Ramanujan — namely the Trinity — should be recognized by SASTRA with honorary doctorates in Ramanujan's hometown, Kumbakonam. The Vice-Chancellor liked this suggestion, and so Andrews, Askey and Berndt were awarded honorary doctorates in a colorful ceremony with traditional Indian music being played as the recipients walked in.

Birthday conferences every five years: Andrews has remained productive defying the passage of time. In view of his enormous influence, and his charm, conferences in his honor have been organized every five years starting from his 60-th birthday, and I have had the privilege of participating in every one of them — in Maratea, Italy in 1998 for his 60-th, in Penn State in 2003 and 2008 for his 65-th and 70-th, in Tianjin, China in 2013 for his 75-th, and now in Penn State for his 80th.

In Indian culture, the 80th birthday is of special significance. Somewhere between the 80th and 81st birthdays, the individual would have seen 1000 crescent moons, and sighting the crescent moon is auspicious in the Hindu religion because it adorns the head of Lord Shiva. George Andrews is still going strong at 80, with no signs of slowing down. So he is full of mathematical energy and continues to be our inspiring leader.

G. H. Hardy once said that he had the unique privilege of collaborating with Ramanujan and Littlewood in something like equal terms. Although I am no Hardy, I can say proudly that I am unique in having had a close collaboration with Paul Erdős and George Andrews, two of the most influential mathematicians of our time! I next describe my joint work with Andrews on the Göllnitz and Capparelli theorems.

3. Collaboration with Andrews

Before describing my joint work with Andrews, I need to briefly provide as background, my joint work with Gordon on Schur's theorem.

One of the first results in the theory of partitions that one encounters, is a lovely theorem of Euler, namely:

Theorem E. *The number of partitions $p_d(n)$ of n into distinct parts, equals the number of partitions $p_o(n)$ of n into odd parts.*

Euler's proof of this was to consider the product generating functions of these two partition functions and show they are equal by using the trick

$$1 + x = \frac{1 - x^2}{1 - x}.$$

More precisely,

$$\sum_{n=0}^{\infty} p_d(n) q^n = \prod_{m=1}^{\infty} (1 + q^m) = \prod_{m=1}^{\infty} \frac{1 - q^{2m}}{1 - q^m} = \prod_{m=1}^{\infty} \frac{1}{1 - q^{2m-1}} = \sum_{n=0}^{\infty} p_o(n) q^n. \qquad (1)$$

Let us think of partitions into distinct parts as those for which the gap between the parts is ≥ 1, and partitions into odd parts as those whose parts are $\equiv \pm 1 \pmod 4$.

If Euler's theorem is viewed in this fashion, then the celebrated Rogers–Ramanujan partition theorem is the "next level" result with gap ≥ 1 replaced by gap ≥ 2 between parts, and the congruence mod 4 replaced by modulus 5. More precisely, the first Rogers–Ramanujan partition theorem is:

Theorem R1. *The number of partitions of an integer n into parts that differ by ≥ 2, equals the number of partitions of n into parts $\equiv \pm 1 \pmod 5$.*

In the second Rogers–Ramanujan partition theorem (R2) we consider partitions whose parts differ by ≥ 2 but do not have 1 as a part, and equate these with partitions into parts $\equiv \pm 2 \pmod 5$. The two Rogers–Ramanujan partition identities can be cast in an analytic form, namely

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{2}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{3}$$

In (2) and (3) and in what follows, we have used the standard notation

$$(a;q)_n = (a)_n = \prod_{j=1}^{n} (1 - aq^{j-1}),$$

and

$$(a)_\infty = \lim_{n\to\infty} (a)_n, \quad \text{for} \quad |q| < 1.$$

When the base is q, then as on the left in (2) and (3), we do not mention it, but when the base is other than q, then we always mention it, as on the right in (2) and (3).

Although the Rogers–Ramanujan identities are the next level identities beyond Euler's theorem, they are much deeper. They also have a rich history that we will not get into here. We just mention that the analytic forms of the identities (2) and (3) were first discovered by Rogers and Ramanujan independently, and it was only later that MacMahon and Schur independently provided the partition version, namely Theorems R1 and R2. Neither Rogers nor Ramanujan mentioned the partition versions of (2) and (3). So in fairness, Theorems R1 and R2 should be called the MacMahon-Schur theorems.

In the theory of partitions and q-series, a Rogers–Ramanujan (R-R) type identity is a q-hypergeometric identity in the form of an infinite (possibly multiple) series equals an infinite product. The series is the generating function of partitions whose parts satisfy certain difference conditions, whereas the product is the generating function of partitions whose parts usually satisfy certain congruence conditions. Since the 1960s, Andrews has spearheaded the study of R-R type identities (see [15], for instance). R-R type identities arise as solutions of models in statistical mechanics as first observed by Rodney Baxter in his fundamental work. After noticing the role of R-R type identities in certain physical problems, Baxter and his group approached Andrews to provide insight into the structure of such identities. Andrews then collaborated with Baxter and Peter Forrester to determine all R-R type identities that arise as solutions of the Hard-Hexagon Model in statistical mechanics. For a discussion of a theory of R-R type identities, see Andrews [15, Ch. 9]. For a discussion of connections with problems in physics, see Andrews' CBMS Lectures [16].

The partition theorem which is the combinatorial interpretation of an R-R type identity, is called a Rogers–Ramanujan type partition identity. A q-hypergeometric R-R type identity is usually discovered first and then its combinatorial interpretation as a partition theorem is given. There are important instances of Rogers–Ramanujan type partition identities being discovered first and their q-hypergeometric versions given later. Perhaps the first such significant example is the 1926 partition theorem of Schur [23].

In emphasizing the partition version of (2) and (3), Schur discovered the "next level" partition theorem, namely:

Theorem S (Schur, 1926). *Let $T(n)$ denote the number of partitions of an integer n into parts* $\equiv \pm 1 \pmod 6$.

Let $S(n)$ denote the number of partitions of n into distinct parts $\equiv \pm 1 \pmod 3$.

Let $S_1(n)$ denote the number of partitions of n into parts that differ by ≥ 3, where the inequality is strict if a part is a multiple of 3. Then

$$T(n) = S(n) = S_1(n).$$

The equality $T(n) = S(n)$ is simple and follows easily by using Euler's trick on their product generating functions, namely

$$\sum_{n=0}^{\infty} T(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty} = (-q;q^3)_\infty (-q^2;q^3)_\infty = \sum_{n=0}^{\infty} S(n)q^n. \tag{4}$$

Thus it is the equality $S(n) = S_1(n)$ which is the real challenge. In 1966, Andrews [11] gave a a new q-theoretic proof of $S(n) = S_1(n)$. This enabled him to discover two infinite families of identities ([12, 13]) modulo $2^k - 1$ emanating from Schur's theorem.

In 1989, in collaboration with Gordon, I obtained a generalization and two parameter refinement of the equality $S(n) = S_1(n)$ (see [10]). The main idea in [10] was to establish the *key identity*

$$\sum_{i,j} a^i b^j \sum_m \frac{q^{T_{i+j-m}+T_m}}{(q)_{i-m}(q)_{j-m}(q)_m} = (-aq)_\infty (-bq)_\infty, \tag{5}$$

and to view a two parameter refinement of the equality $S(n) = S_1(n)$ as emerging from (5) under the transformations

$$(\textit{dilation}) \quad q \mapsto q^3, \quad \text{and} \quad (\textit{translations}) \quad a \mapsto aq^{-2}, b \mapsto bq^{-1}. \tag{6}$$

In (5) and below, $T_m = m(m+1)/2$ is the m-th triangular number.

The interpretation of the product in (5) as the generating function of bi-partitions into distinct parts in two colors is clear. In [10] it was shown that the series in (5) is the generating function of partitions (= words with weights attached) into distinct parts occurring in three colors - two primary colors a and b, and one secondary color ab, and satisfying certain gap conditions. We describe this now.

We assume that the integer 1 occurs in two primary colors a and b, and that each integer $n \geq 2$ occurs in the two primary colors as well as in the secondary color ab. By a_n, b_n, and ab_n, we denote the integer n in colors a, b, and ab respectively. In order to discuss partitions, we need to impose an order on the colors, and the order that Gordon and I chose is

$$a_1 < b_1 < ab_2 < a_2 < b_2 < ab_3 < a_3 < b_3 < \cdots . \tag{7}$$

Thus for a given integer n, the order of the colors is

$$ab < a < b. \tag{8}$$

The transformations in (6) correspond to the replacements

$$a_n \mapsto 3n-2, \quad b_n \mapsto 3n-1, \quad \text{and} \quad ab_n \mapsto 3n-3, \tag{9}$$

Under (9), the ordering of the colored integers in (7) becomes

$$1 < 2 < 3 < 4 \cdots ,$$

the standard ordering among the positive integers. This is one of the reasons Gordon and I chose the ordering in (7).

Using the colored integers, Gordon and I gave the following partition interpretation for the series in (5). We defined *Type 1* partitions as those of the form $x_1+x_2+\cdots$, where the x_i are symbols from the sequence in (7) with the condition that the *gap* between x_i and x_{i+1}, namely the difference between the subscripts of the colored integers they represent, is ≥ 1, with strict inequality if

$$x_i \quad \text{has a lower order color compared to} \quad x_{i+1}, \tag{10a}$$

or

$$x_i, \quad x_{i+1} \quad \text{are both of secondary color.} \tag{10b}$$

In (10a), the order of the colors is as in (8).

Using (9) it can be shown that that the gap conditions of Type 1 partitions in (10a) and (10b) translate to the difference conditions of $S_1(n)$ in Schur's theorem. Two proofs of (5) were given in [10] — one combinatorial, and another using the q-Chu–Vandermonde Summation. Thus the R-R type identity for Schur's theorem came half a century later.

Gordon then suggested that we should apply the method of weighted words to generalize and refine the deep 1967 theorem of Göllnitz [19] which is:

Theorem G. *Let $B(n)$ denote the number of partitions of n into parts $\equiv 2$, 5, or $11 \pmod{12}$.*

Let $C(n)$ denote the number of partitions of n into distinct parts $\equiv 2$, 4, or $5 \pmod 6$.

Let $D(n)$ denote the number of partitions of n into parts that differ by ≥ 6, where the inequality is strict if a part is $\equiv 0$, 1, or $3 \pmod 6$, and with 1 and 3 not occurring as parts. Then

$$B(n) = C(n) = D(n).$$

The equality $B(n) = C(n)$ is easy because

$$\begin{aligned}\sum_{n=0}^{\infty} B(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{12m-10})(1-q^{12m-7})(1-q^{12m-1})} \\ &= \prod_{m=1}^{\infty}(1+q^{6m-4})(1+q^{6m-2})(1+q^{6m-1}) = \sum_{n=0}^{\infty} C(n)q^n. \end{aligned} \tag{11}$$

This is one reason that we focus on the deeper equality $C(n) = D(n)$, the second reason being that it is this equality which can be refined.

Göllnitz' proof of Theorem G is very intricate and difficult but he succeeded in proving Theorem G in the refined form

$$C(n;k) = D(n;k), \tag{12}$$

where $C(n;k)$ and $D(n;k)$ denote the number of partitions of the type counted by $C(n)$ and $D(n)$ respectively, with the extra condition that the number of parts is k, and with the convention that parts $\equiv 0$, 1, or 3 (mod 6) are counted twice. Andrews [14] subsequently provided a simpler proof. I think besides Göllnitz, Andrews is the only other person to have gone through the difficult details of Göllnitz' proof of Theorem G. In Chapter 10 of his famous CBMS Lectures [16], Andrews asks for a proof that will provide insights into the structure of the Göllnitz theorem.

In view of (12) and our work on Schur's theorem, Gordon suggested that we should look at Göllnitz' theorem in the context of the method of weighted words. To this end, Gordon and I first considered the product

$$(-aq)_\infty(-bq)_\infty(-cq)_\infty \tag{13}$$

and viewed the generating function of $C(n)$ as emerging out of (13) under the substitutions

$$(\textit{dilation}) \quad q \mapsto q^6, \quad \text{and } (\textit{translations}) \quad a \mapsto aq^{-4}, b \mapsto bq^{-2}, c \mapsto cq^{-1}. \tag{14}$$

The problem then was to find a series that would sum to this product, with the series representing the generating function of partitions into colored integers with gap conditions that would correspond to those governing $D(n)$. What Gordon and I did was to consider the integer 1 to occur in three primary colors a, b, and c, and integers $n \geq 2$ to occur in these three primary colors as well as in three secondary colors ab, ac, and bc. As before, the symbols $a_n, b_n, \cdots, bc_n$ represent n in colors $a, b, \cdots, bc$ respectively. Here too we need an ordering on the colored integers, and the one we chose is

$$a_1 < b_1 < c_1 < ab_2 < ac_2 < a_2 < bc_2 < b_2 < c_2 < ab_3 < \ldots. \tag{15}$$

The effect of the substitutions (14) is to convert the symbols to

$$\begin{cases} a_m \mapsto 6m-4,\ b_m \mapsto 6m-2,\ c_n \mapsto 6m-1, & \text{for}\, m \geq 1, \\ ab_m \mapsto 6m-6,\ ac_m \mapsto 6m-5,\ bc_n \mapsto 6m-3, & \text{for}\, m \geq 2. \end{cases} \tag{16}$$

so that the ordering (15) becomes

$$2 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12 < \cdots, \tag{17}$$

This is one reason for the choice of the ordering of symbols in (15), because they convert to the natural ordering of the integers in (17) under the transformations (16). Notice that 1, and 3 are missing in (17), and this explains the condition that 1 and 3 do not occur as parts in the partitions enumerated by $D(n)$ in Theorem G.

To view Theorem G in this context, we think of the primary colors a, b, c as corresponding to the residue classes 2, 4 and 5 (mod 6) and so the secondary colors ab, ac, bc correspond to the residue classes $2+4 \equiv 6$, $2+5 \equiv 7$ and $4+5 \equiv 9$ (mod 6). Note that integers of secondary color occur only when $n \geq 2$ and so ab_1, ac_1 and bc_1 are missing in (15). This is why integers $ac_1 = 1$ and $bc_1 = 3$ do not appear in (17). This explains the absence of 1 and 3 among the parts enumerated by $D(n)$ in Theorem G. Note that ab_1 corresponds to the integer 0, which is not counted as a part in ordinary partitions anyway.

In (15) for a given subscript, the ordering of the colors is

$$ab < ac < a < bc < b < c. \tag{18}$$

We use (18) to say for instance that ab is of *lower order* compared to a, or equivalently that a is of *higher order* than ab. With this concept of the order of colors, we can define *Type 1* partitions to be of the form $x_1 + x_2 + \ldots$, where the x_i are symbols from (15) with the condition that the gap between x_i and x_{i+1} is ≥ 1 with strict inequality if

$$x_i \quad \text{is of lower order (color) compared to} \quad x_{i+1}, \tag{19a}$$

or

$$\text{if} \quad x_i \quad \text{and} \quad x_{i+1} \quad \text{are of the same secondary color.} \tag{19b}$$

Under the transformations given by (16), the gap conditions of Type 1 partitions become the difference conditions governing $D(n)$. Gordon and I then showed that the generating function of Type 1 partitions is

$$\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} \tag{20}$$

Thus our three three parameter *key identity* for the generalization and refinement of Göllnitz' theorem is

$$\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi}$$
$$= \sum_{i,j,k} \frac{a^i b^j c^k q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k} = (-aq)_\infty(-bq)_\infty(-cq)_\infty\ , \tag{21}$$

The partition interpretation of (21) that Gordon and I had was:

Theorem 1. *Let $C(n; i, j, k)$ denote the number of vector partitions $(\pi_1; \pi_2; \pi_3)$ of n such that π_1 has i distinct parts all in color a, π_2 has j distinct parts all in color b, and π_3 has k distinct parts all in color c.*

Let $D(n; \alpha, \beta, \gamma, \delta, \varepsilon, \phi)$ denote the number of Type 1 partitions of n having α a-parts, β b-parts, $\ldots$, and ϕ bc-parts.

Then

$$C(n; i, j, k) = \sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} D(n; \alpha, \beta, \gamma, \delta, \varepsilon, \phi).$$

It is to be noted that in Theorem 1,

$$i + j + k = \alpha + \beta + \gamma + 2(\delta + \epsilon + \phi)$$

and so the parts in secondary color are counted twice. This corresponds to the condition that parts $\equiv 0, 1, 3 \pmod 6$ are counted twice in (12).

The proof in [8] that the expression in (20) is the generating function of minimal partitions is quite involved and goes by induction on $s = \alpha + \beta + \gamma + \delta + \varepsilon + \phi$, the number of parts of the Type-1 partitions, and also appeals to *minimal partitions* whose generating functions are given by multinomial coefficients (see [8] for details).

Thus everything fitted perfectly, but Gordon and I had a problem: *we could not prove the key identity* (21). This is where Andrews entered into the picture. The story of how he proved (21) is described in Part 1. His ingenious proof of the remarkable key identity (21) relied on the Watson's q-analogue of Whipple's transformation and the ${}_6\psi_6$ summation of Bailey. For the proof of (21), we refer the reader to [8]. Let me just say, that there is no one in the world who can match Andrews' power in proving multi-variable q-hypergeometric identities!

One of the great advantages of the method of weighted words is that it provides a key identity for a partition theorem at the base level, and from this one can extract several partition theorems by suitable dilations and translations. I investigated in detail a variety of partition theorems that emerge from (21) (see [1, 3]), but will report here only two major developments that involved Andrews.

As noted earlier, Göllnitz' theorem pertains to the dilation $q \mapsto q^6$ in (21), and so I wanted to investigate the effect under the transformations

$$(\textit{dilation}) \quad q \mapsto q^3, \tag{22a}$$

and

$$(\textit{translations}) \quad a \mapsto aq^{-2}, \quad b \mapsto bq^{-1}, \quad c \mapsto c. \tag{22b}$$

In this case the product in (21) becomes

$$\prod_{m=1}^{\infty}(1+aq^{3m-2})(1+bq^{3m-1})(1+q^{3m}),$$

which is the three parameter generating function of partitions into distinct parts, and therefore is very interesting. The dilation $q \mapsto q^6$ converts the six colors $a, b, \cdots, bc$ into the six different residue classes mod 6, and under the dilation in (22a), one gets partitions into parts that differ by ≥ 3 but these partitions have to be counted with a weight because each positive integer ≥ 3 occurs in two colors - one primary and one secondary. Two major consequences of this weighted partition identity were (i) a new proof of Jacobi's triple product identity for theta functions, and (ii) a combinatorial proof of a variant of Göllnitz' theorem which is equivalent to it. In the course of identifying this variant, I found a new *cubic key identity* that represents it, namely

$$\sum_{i,j,k} \frac{a^i b^j c^k (-c)_i (-c)_j (-\frac{ab}{c}q)_k (-cq)_{i+j} q^{T_{i+j+k}}}{(q)_i (q)_j (q)_k (-c)_{i+j}} = (-aq)_\infty (-bq)_\infty (-cq)_\infty. \tag{23}$$

As in the case of (21), I approached Andrews for a proof of (23), and he supplied it in a matter of a few days utilizing Jackson's q-analogue of Dougall's summation. This led to our second joint paper [4]. While (23) is quite deep, it is simpler in structure compared to (21).

Next I investigated the combinatorial consequences of (21) under the

$$(\textit{dilation}) \quad q \mapsto q^4, \tag{24a}$$

but here there are four possible translations depending on which residue class modulo 4 one chooses to omit for the primary color. For example, the translations

$$a \mapsto aq^{-3}, \quad b \mapsto bq^{-1}, \quad c \mapsto cq^{-3}, \tag{24b}$$

omits the residue class 0 (mod 4) for the primary colors, and there are three other important dilations. Some very interesting weighted partition identities emerge (see

[3]), but I focused on the translations in (24b) owing to the symmetry. This led me to the following *quartic key identity*:

$$\sum_{i,j,k,\ell} \frac{a^{i+\ell}b^j c^{k+\ell} q^{T_{i+j+k+\ell}+T_\ell} \left(-\frac{bc}{a}\right)_i \left(-\frac{abq}{c}\right)_k}{(q)_i(q)_j(q)_k(q)_\ell} \frac{(1+\frac{bc}{a}q^{2i-1})}{(1+\frac{bc}{a}q^{i-1})} = (-aq)_\infty(-bq)_\infty(-cq)_\infty, \tag{25}$$

Once again, I approached Andrews for a proof of (25), and he supplied it using Jackson's q-analogue of Dougall's summation. This led to my third paper with Andrews [5].

When Göllnitz proved his theorem in 1967, it was viewed as a next level result beyond Schur's theorem because the two residue classes $1, 2 \pmod 3$ for $S(n)$ in Schur's Theorem are replaced by three residue classes 2, 4, 5 (mod 6) for $C(n)$ in Göllnitz' theorem. Apart from this, it is not clear why Göllnitz' theorem can be considered as an extension of Schur's. But then, by our method of weighted words, one sees exactly how our generalized Göllnitz Theorem 1 is an extension of Schur's to the next level, because the key identity (5) for Schur's theorem is simply the special case $c = 0$ in the key identity (21) for Göllnitz' theorem.

So if Göllnitz' theorem is the "next level" result beyond Schur's theorem, why is it so much more difficult to prove? One reason for this is because in Göllnitz' theorem, when expanding the product in (21), we consider only the primary and secondary colors in the series and omit the ternary color abc. Actually, as early as 1968 and 69, Andrews [12, 13], had obtained two infinite hierarchies of partition theorems to moduli $2^k - 1$ when $k \geq 2$, where he starts with k residue classes (mod $2^k - 1$) and considers the complete set of residue classes (mod $2^k - 1$) for the difference conditions. We now describe his results:

For a given integer $r \geq 2$, let $a_1, a_2, \ldots, a_r$ be r distinct positive integers such that

$$\sum_{i=1}^{k-1} a_i < a_k, \quad 1 \leq k \leq r. \tag{26}$$

Condition (26) ensures that the $2^r - 1$ sums $\sum \varepsilon_i a_i$, where $\varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0$, are all distinct. Let these sums in increasing order be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_{2^r-1}$.

Next let $N \geq \sum_{i=1}^r a_i \geq 2^r - 1$ be a modulus, and A_N denote the set of all positive integers congruent to some $a_i \pmod N$. Similarly, letA'_N denote the set of all positive integers congruent to some $\alpha_i \pmod N$ Also let $\beta_N(m)$ denote the least positive residue of $m \pmod N$. Finally, if $m = \alpha_j$ for some j, let $\phi(m)$ denote the number of terms appearing in the defining sum of m and $\psi(m)$ the smallest a_i appearing in this sum. Then the first general theorem of Andrews [12] is:

Theorem A1. *Let $C^*(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N.*

Let $D^*(A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, \ldots, b_\nu$ from A'_N such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(b_{i+1})) + \psi(\beta_N(b_{i+1})) - \beta_N(b_{i+1}). \tag{27}$$

Then

$$C^*(A_N; n) = D^*(A'_N; n).$$

To describe the second general theorem of Andrews (1969), let a_i, α_i and N be as above. Now let $-A_N$ denote the set of all positive integers congruent to some

$-a_i \pmod N$), and $-A'_N$ the set of all positive integers congruent to some $-\alpha_i \pmod N$). The quantities $\beta_N(m), \phi(m), \psi(m)$ are also as above. We then have (Andrews [13]):

Theorem A2. *Let $C(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$.*

Let $D(-A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, \ldots, b_\nu$, taken from $-A'_N$ such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(-b_i)) + \psi(\beta_N(-b_i)) - \beta_N(-b_i) \tag{28}$$

and also

$$b_\nu \geq N(\phi(\beta_N(-b_s) - 1)).$$

Then

$$C(-A_N; n) = D(-A'_N; n).$$

When $r = 2$, $a_1 = 1$, $a_2 = 2$, $N = 3 = 2^r - 1$, Theorems A1 and A2 both become Theorem S. Thus the two hierarchies emanate from Theorem S, and it is only when $r = 2$ that the hierarchies coincide. Thus Theorem S is its own dual. Conditions (27) and (28) can be understood better by classifying b_{i+1} (in Theorem A1) and b_i (in Theorem A2) in terms of their residue classes (mod N). In particular, with $r = 3$, $a_1 = 1$, a_2, $a_3 = 4$ and $N = 7 = 2^3 - 1$, Theorems A1 and A2 yield the following corollaries.

Corollary 1. *Let $C^*(n)$ denote the number of partitions of n into distinct parts $\equiv 1$, 2 or $4 \pmod 7$.*

Let $D^*(n)$ denote the number of partitions of n in the form $b_1 + b_2 + \cdots_\nu$ such that $b_i - b_{i+1} \geq 7, 7, 12, 7, 10, 10$ or 15 if $b_{i+1} \equiv 1, 2, 3, 4, 5, 6$ or $7 \pmod 7$. Then

$$C^*(n) = D^*(n).$$

Corollary 2. *Let $C(n)$ denote the number of partitions of n into distinct parts $\equiv 3$, 5 or $6 \pmod 7$.*

Let $D(n)$ denote the number of partitions of n in the form $b_1 + b_2 + \cdots + b_\nu$ such that $b_i - b_{i+1} \geq 10, 10, 7, 12, 7, 7$ or 15 if $b_i \equiv 8, 9, 3, 11, 5, 6$ or $14 \pmod 7$ and $b_\nu \neq 1, 2, 4$ or 7. Then

$$C(n) = D(n).$$

Andrews' proofs of Theorems A1 and A2 are extensions of his proof [11] of Theorem S and not as difficult as the proof of Göllnitz' theorem. During the 1998 conference in Maratea, Italy, for Andrews' 60-th birthday organized by Dominique Foata, I gave a talk outlining a method of weighted words approach generalization of Theorems A1 and A2 that Gordon and I had worked out. In our approach, we obtain an "amalgamation process" that yields the weighted words generalization of the Andrews hierarchy without a q-hypergeometric key identity. In our approach we have a single partition theorem with multiple parameters, and the two Andrews hierarchies turn out to be the two extreme (special) cases under suitable choices of the parameters (for a description of the main ideas of this approach to the Andrews hierarchies, see [2], p. 25-27). Dominique Foata then asked whether there is a hypergeometric key identity that corresponds to this generalization. Even though the proofs of Theorems A1 and A2 are simpler compared to the the proof of Theorem G, no hypergeometric key identity has yet been found to represent the Andrews hierarchies when $2^k - 1 > 3$.

In view of the fact that with a complete set of alphabets one gets an infinite hierarchy of theorems, Andrews raised as a problem in his CBMS Lectures, whether there exists a partition theorem beyond Göllnitz' theorem in the same manner as Göllnitz' theorem goes beyond Schur. In the language of the method of weighted words, this is the same as asking whether there exists a partition theorem starting with four primary colors a, b, c, d and using only a proper subset of the complete alphabet of 15 colors, that will yield Göllnitz' theorem when we set the parameter $d = 0$. The answer to this difficult problem was found by Alladi-Andrews-Berkovich in 2000, by noticing that ALL ternary colors have to be dropped but the quaternary color $abcd$ needs to be retained. This led to a remarkable identity in four parameters a, b, c, d that went beyond (21). Our paper [7] describes the ideas behind the construction of this four parameter identity, and provides the proof as well. I just mention here a striking (mod 15) identity that emerges from this four parameter q-hypergeometric identity:

Theorem AAB. *Let $P(n)$ denote the number of partitions of n into distinct parts* $\equiv -2^3, -2^2, -2^1, -2^0 \pmod{15}$.

Let $G(n)$ denote the number of partitions of n into parts $\not\equiv 2^0, 2^1, 2^2, 2^3 \pmod{15}$, *such that the difference between the parts is ≥ 15, with equality only if a part is* $\equiv -2^3, -2^2, -2^1, -2^0 \pmod{15}$, *parts which are* $\equiv \pm 2^0, \pm 2^1, \pm 2^2, \pm 2^3 \pmod{15}$ *are* > 15, *the difference between the multiples of* 15 *is* ≥ 60, *and the smallest multiple of* 15 *is*

$$\begin{cases} \geq 30 + 30\tau, & \text{if } 7 \text{ is a part, and} \\ \geq 45 + 30\tau, & \text{otherwise,} \end{cases}$$

where τ is number of non–multiples of 15 *in the partition. Then*

$$G(n) = P(n).$$

One aspect of Göllnitz' Theorem G that escaped attention was whether it had a dual in the sense that Theorems A1 and A2 can be considered as duals. More precisely, the residue classes of Corollary 1 that constitute the primary colors are $1, 2, 4 \pmod 7$, whereas the residue classes that constitute the primary colors in Corollary 2 are $-1, -2, -4 \pmod 7$. Now one can view $2, 4, 5 \pmod 6$ as $-1, -2, -4 \pmod 6$. So the question is whether there is a dual result to Theorem G starting with $1, 2, 4 \pmod 6$. Andrews found such a theorem, namely:

Theorem A. *Let $B^*(n)$ denote the number of partitions of n into parts* $\equiv 1$, 7, *or* $10 \pmod{12}$.

Let $C^*(n)$ denote the number of partitions of n into distinct parts $\equiv 1$, 2, *or* $4 \pmod 6$.

Let $D^*(n)$ denote the number of partitions of n into parts that differ by at least 6, *where the inequality is strict if the larger part is* $\equiv 0$, 3, *or* $5 \pmod 6$, *with the exception that* $6 + 1$ *may appear in the partition. Then*

$$B^*(n) = C^*(n) = D^*(n).$$

Andrews provided a proof of Theorem A very similar to his proof of Theorem G in [14]. My role then was to construct a key identity that represented this dual, which I did. This key identity for the dual, although different from (21), is equivalent to it. This led to our most recent joint paper [6] which we published in a Special Volume of The Ramanujan Journal dedicated to the memory of Basil Gordon.

I conclude by describing my joint paper with Andrews on the Capparelli partition theorems.

In fundamental work [20, 21], Lepowsky and Wilson gave a Lie theoretic proof of the Rogers–Ramanujan identities and in that process showed how R-R type identities arise in the study of vertex operators in Lie algebras. Using vertex operator theory, Stefano Capparelli, a PhD student of Lepowsky in 1992, “discovered” two new partition results [18] which he could not prove and so he stated them as conjectures:

Conjecture C1. *Let $C^*(n)$ denote the number of partitions of n into parts $\equiv \pm 2$, $\pm 3 \pmod{12}$.*

Let $D(n)$ denote the number of partitions of n into parts > 1 with minimal difference 2, where the difference is ≥ 4 unless consecutive parts are both multiples of 3 or add up to a multiple of 6. Then

$$C^*(n) = D(n).$$

He had a second partition result, **Conjecture C2**, which we do not state here because the conditions are more complicated; also that is not essential to what we will describe here.

As mentioned in Part I, Lepowsky stated Conjecture C1 on the opening day of the Rademacher Centenary Conference at Penn State, and by the time that conference ended, Andrews had a proof using q-recurrences (see [17]).

The first thing I did on seeing Conjecture C1 was to replace $C^*(n)$ by $C(n)$, the number of partitions of n into distinct parts $\equiv 2, 3, 4$ or $6 \pmod 6$, and to note that

$$C(n) = C^*(n) \tag{29}$$

This is because by Euler's trick

$$\sum_{n=0}^{\infty} C^*(n) q^n = \frac{1}{(q^2;q^{12})_\infty (q^3;q^{12})_\infty (q^9;q^{12})_\infty (q^{10};q^{12})_\infty}$$

$$= (-q^2;q^6)_\infty (-q^4;q^6)_\infty (-q^3;q^3)_\infty = \sum_{n=0}^{\infty} C(n) q^n. \tag{30}$$

One reason for replacing $C^*(n)$ by $C(n)$ is that the equality in (29) can be refined. Another reason is that Conjecture C2 can be more elegantly stated by replacing $C(n)$ by the function $C'(n)$ which enumerates the number of partitions into distinct parts $\equiv 1, 3, 5,$ or $6 \pmod 6$.

The refinement of the Capparelli Conjecture C1 that Andrews, Gordon and I [9] proved was:

Theorem 2. *Let $C(n; i, j, k)$ denote the number of partitions counted by $C(n)$ with the additional restriction that there are precisely i parts $\equiv 4 \pmod 6$, j parts $\equiv 2 \pmod 6$, and of those $\equiv 0 \pmod 3$, exactly k are $> 3(i + j)$.*

Let $D(n; i, j, k)$ denote the number of partitions counted by $D(n)$ with the additional restriction that there are precisely i parts $\equiv 1 \pmod 3$, j parts $\equiv 2 \pmod 3$, and k parts $\equiv 0 \pmod 3$. Then

$$C(n; i, j, k) = D(n; i, j, k).$$

To establish Theorem 2, we put it in the context of the method of weighted words. More precisely, let the integer 1 occur in two colors a and c and let integers ≥ 2 occur in three colors a, b and c. As before, the symbols a_j, b_j and c_j represent the integer j in colors a, b and c respectively. To discuss partitions the ordering of the symbols we used is

$$a_1 < b_2 < c_1 < a_2 < b_3 < c_2 < a_3 < b_4 < c_3 < \cdots . \tag{31}$$

The Capparelli problem corresponds to the transformations

$$a_j \mapsto 3j-2, \qquad b_j \mapsto 3j-4, \qquad c_j \mapsto 3j, \tag{32}$$

in which case the inequalities in (31) become

$$1 < 2 < 3 < 4 < 5 < \cdots ,$$

the natural ordering among the positive integers. With this we were able to generalize and refine Theorem 2 as follows:

Theorem 3. *Let $K(n; i, j, k)$ denote the number of vector partitions of n in the form (π_1, π_2, π_3) such that π_1 has distinct even a-parts, π_2 has distinct even b-parts, and π_3 has distinct c-parts such that $\nu(\pi_1) = i$, $\nu(\pi_2) = j$, and the number of parts of π_3 which are $> i + j$ is k.*

Let $G(n; i, j, k)$ denote the number of partitions (words) of n into symbols a_j, b_j, c_j each $> a_1$, such that the gap between consecutive symbols is given by the matrix below:

	a	b	c
a	2	2	1
b	0	2	0
c	2	3	1

Then

$$K(n; i, j, k) = G(n; i, j, k).$$

Note. The matrix above is to read row-wise. Thus if a_j is a part of the partition, and the next larger part has color b, then its weight (= subscript) must be $> j + 2$.

In [9] we gave a combinatorial proof of Theorem 2 by using some ideas of Bressoud, and another proof by first showing that it is equivalent to the following *key identity*

$$\begin{aligned}\sum_{i,j,k,n} K(n; i, j, k) a^i b^j c^k q^n &= \sum_{i,j} \frac{a^i b^j q^{2T_i+2T_j}(-q)_{i+j}(-cq^{i+j+1})_\infty}{(q^2; q^2)_i (q^2; q^2)_j} \\ = \sum_{i,j,k,n} G(n; i, j, k) a^i b^j c^k q^n &= \sum_{i,j,k} \frac{a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}} \times \begin{bmatrix} i+j+k \\ i+j, k \end{bmatrix}_q \begin{bmatrix} i+j \\ i, j \end{bmatrix}_{q^2},\end{aligned} \tag{33}$$

and then proving this identity.

The main difficulty in (33) was to show that the series on the right is the generating function of partitions with gap conditions given by the entries in the above table. This required the study of minimal partitions having a part in a specified color as the smallest part. Once the generating function of the $G(n; i, j, k)$ was shown to be the series on the right in (29), it was not difficult to establish the equality of this with the series on the left. If we take $c = 1$, then the generating function on the left in (33) becomes a product, because

$$(-q)_\infty \sum_{i,j,k} \frac{a^i b^j q^{2T_i+2T_j}}{(q^2;q^2)_i (q^2;q^2)_j} = (-q)_\infty (-aq^2;q^2)_\infty (-bq^2;q^2)_\infty. \tag{34}$$

In (30) if we replace $q \mapsto q^3$, $a \mapsto q^{-2}$, $b \mapsto q^{-4}$, we get

$$\prod_{j=0}^{\infty} (1+q^{6j-2})(1+q^{6j-4})(1+q^{3j}) = \sum_{n=0}^{\infty} C(n) q^n,$$

and so Capparelli's conjecture follows.

I could say so much more about Andrews' work on partitions, q-series and Ramanujan, but here I chose to focus on an aspect of our joint work that shows that in manipulating q-hypergeometric series, he has no match in our generation. Even though he towers head and shoulders above the rest in the world of partitions, q-series and Ramanujan, he is a perfect gentleman always willing to help. It is a pleasure and a privilege for me to be his friend and collaborator.

Acknowledgements. It was a privilege to have been an organizer of the Andrews80 Conference, but I have to thank the other organizers of the Conference - Bruce Berndt, Peter Paule, James Sellers, and Ae Ja Yee - not only to have invited me to be the Opening Speaker, but also to have given me an opportunity to speak at the Banquet when I recalled my long and close association with George.

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