NEILS HENRIK ABEL

Norwegian mathematical genius

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Editor's note: December 22, 2004, is Srinivasa Ramanujan's 117-th birth anniversary. To mark the occasion, Krishnaswami Alladi writes about the life and contributions of the Norwegian mathematician Abel and makes comparisons with the Indian genius Ramanujan.

Neils Henrik Abel, the great Norwegian mathematical genius, lived only for twenty seven years. Like Ramanujan, Abel fought poverty and other hardships, and in his brief life made path breaking contributions to several branches of mathematics. Abel and Jacobi, simultaneously and independently, discovered the elliptic functions while considering the inversion of certain integrals. Abel was the first to establish rigorously that there is no general formula to solve all polynomial equations of degree five; subsequently Galois established the impossibility of the solvability of polynomials of degree at least five, and this led to the creation of group theory. In the course of his research Abel realised the need for a thorough treatment of infinite series. He not only obtained several important results on convergence of series, but also provided a rigorous basis for mathematical analysis, an area that was shaped subsequently into a solid mathematical theory by the French mathematician Cauchy. Abel's contributions place him among the greatest mathematicians in history. In 2002, on the occasion of the bicentennial of his birth, the Norwegian Academy of Science and Letters, established the Abel Prize for Mathematics, matching the Nobel Prize in prestige and prize money. In this article I shall decribe the life and contributions of Abel and make comparisons with the life and work of Ramanujan. Toward the end I will also describe briefly the purpose and need of the Abel Prize and what led to its creation.

Early life: Neil Henrik Abel was born on 5 August 1802 as the second child of Soren Georg Abel and Anne Marie Simonsen. Soren was a vicar. The family was living at the Finnoy vicarage in western Norway. Later the family moved to Gjerstad in Southern Norway where Abel grew up with five siblings. Although Sorenan was outstanding parishioner, he lived during a period of severe economic crisis in Norway and thus was unable to provide the financial support his family needed. There were various political conflicts at that time involving England, France, Denmark, Sweden and Norway, as a consequence of which the continental powers blockaded England which countered by blockading Norway. The twin blockade was disastrous to Norway which relied on timber exports to England and grain imports from Denmark, and this precipitated the economic crisis. Thus Abel's life was dominated by poverty. Throughout history, we see examples of geniuses who have made outstanding discoveries in spite of financial difficulties that plagued their lives. To such gifted minds like Abel and Ramanujan, the pursuit of knowledge was the primary goal, and they did not let such hardships stand in the way of their intellectual progress.

Exposure to mathematics: In 1815, at the age of 13, Abel was sent to the Cathedral School in Christiana (now Oslo). Although this was a good school, it was in a bad state during the time Abel joined. In this uninspiring atmosphere, Abel initially did not show any talent for mathematics. Educational reforms in Norway in the early nineteenth

century replaced the traditional arrangement of one teacher dealing with all subjects for a certain class, by teachers who taught only certain subjects. Unfortunately, the school's mathematics teacher was very old fashioned in his teaching methods; he would insist on students copying the material from the blackboard, and box the ears of any pupil who did not understand the subject. In one instance, he punished a pupil so severely, that the poor student died! The mathematics teacher was immediately dismissed and a new mathematics teacher Bernt Holmboe joined the school in 1817. This was a welcome change for Abel, because Holmboe began to give individual projects to his students. Holmboe soon discovered the immense talent in sixteen year old Abel, and started tutoring him privately. Thus within a year of Holmboe's arrival, Abel was reading the works of Newton, Euler, Lagrange and Laplace. Unfortunately, tragedy struck Abel's family at this time; Abel's father died in 1820. Thus there was no money to support Abel in school, and on top of this Abel had to shoulder the responsibility of supporting his mother and the family.

Fortunately, with Holmboe's help, Abel was able to obtain a scholarship to complete his school education and in 1821 joined the University of Christiana which was founded just ten years before. The university offered studies in theology, medicine, and law, but nothing at that time in the natural sciences. Thus Abel pursued mathematics on his own by borrowing books from the university library. Thus he could pursue his investigations on the quintic equation, a study that he had begun during his final year at school.

Unsolvability of the general quintic: In high school we are taught the quadratic formula which provides the roots of any quadratic equation in terms of radicals involving the coefficients of the equation. We are told that there are similar formulae for roots of cubic and quartic polynomial equations, but these are more complicated, and so, in school we are not taught these formula. It turns out that there is no such general formula that will apply to all quintic equations. Many mathematicians prior to Abel had unsuccessfully tried to resolve this problem, but it was Abel who finally proved the unsolvability of the general quintic. Actually, in 1821 Abel first announced his result on quintics in a paper submitted to the Danish mathematician Ferdinand Degen for publication in the Royal Society of Copenhagen. When Degen asked Abel to supply a numerical example in support of the method, Abel found an error in his argument. Abel later fixed the mistake in his proof and in 1823 published a few papers on the quintic and related topics in Norway's first mathematics jounal *Magazin for Naturvidenskaberne*.

Although Abel had visited Copenhagen in the interim period, some of the professors who were providing support for Abel realised that he needed to visit the great centres in Paris and Göttingen and come in contact with eminent mathematicians like Gauss and Cauchy. For this reason he received a government grant but was deemed too immature to travel abroad rightaway. In preparation for his trip abroad, Abel arranged at his own expense the publication of a short paper devoted just to the quintic. The problem, as Abel would realise later was, that the paper was too condensed and so it did not attract the attention it deserved. Subsequently he published an expanded version on this topic about which he said "Mathematicians have been very much absorbed with finding the general solution of algebraic equations and several of them have tried to prove the impossibility of it. However, if I am not mistaken, they have not as yet succeeded. I therefore dare hope that mathematicians will receive this memoir with good will, for its purpose to fill this gap

in the theory of algebraic equations."

Ramanujan and radicals: In the conventional school education in India, we are asked to do some problems involving surds (radicals). Ramanujan was a master at identities involving radicals, and made some startling evaluations. There is a charming article on Ramanujan and radicals by Bruce Berndt, H. H. Chan, and L. C. Zhang that appeared about seven years ago in which some of Ramanujan's identities involving radicals are discussed. It is to be noted that Ramanujan was interested in studying equations that can be solved and finding algorithms and relations involving the solutions. He was not interested in proving that certain equations did not have a formula for solutions, or did not have solutions of a certain type. Thus, the unsolvability of the quintic and higher degree polynomials, or Fermat's Last Theorem, did not attract Ramanujan's attention.

Discovery of elliptic functions: Abel sent his six page paper on the quintic to several mathematicians including Gauss who he intended to visit. He obtained a scholarship in 1825 for the purpose of visiting Göttingen and Paris. Abel arrived first in Copenhagen and was distressed to find that Degen had died. In Copenhagen he was given a letter of introduction to Leopold von Crelle of Berlin who had plans to launch a new periodical *Journal für die Reine und Angewandte Mathematik* (Journal for the Pure and Applied Mathematics). Abel was expecting a reply from Gauss, but that never came. Two reasons for Gauss' silence are given. The first is that Gauss had himself proved the impossibility of solving the quintic and was willing to let Abel take the credit. The second was that Gauss was not interested in solvability of equations by radicals, since he had expressed in his 1801 thesis that such an approach is no better than devising a symbol for the solution and declaring that the equation has a root equal to that symbol! In any case, Gauss' silence disappointed Abel, who changed his plans and went to Berlin instead to meet Crelle.

Abel's contact with Crelle was one of the most fortituous events of his life. Abel at that time was studying certain algebraic integrals and had a profound realization that one really ought to investigate their inverse functions. These turned out to be doubly periodic functions and are called elliptic functions. The integrals themselves are called elliptic integrals, because the expression of the arc length of an ellipse is one such integral. Jacobi, simultaneously and independently, had come to the same realisation, and so Abel and Jacobi are the founders of the theory of elliptic functions. Abel wrote a series of fundamental papers on elliptic functions and integrals which were published in the first few issues of Crelle's journal. In fact, the very first issue of Crelle's journal which appeared at the dawn of 1826, contained seven papers of Abel! Thanks in a large part to Abel's papers, Crelle's journal became very soon one of Europe's leading journals. Indeed, Crelle's journal is one of the top mathematics journals even today, and is perhaps the oldest mathematics journal among all those that are now in publication.

Ramanujan and elliptic functions: In studying the works of Ramanujan, one is always led to the following question: How much did Ramanujan know about a particular topic and how much of it did he create himself? Ramanujan obtained several marvellous results on elliptic functions and so the question of whether he had access to books in India on this topic is of interest. As Hardy said, Ramanujan never claimed to have invented elliptic functions. Hardy felt that Ramanujan must have learnt the basics of elliptic function theory from books at libraries in Madras, such as the one due to Greenhill. However, Bruce

Berndt of the University of Illinois, who has thoroughly analysed Ramanujan's notebooks and edited them in five volumes, holds the following view: Although there is evidence that Ramanujan was acquainted with the books of Greenhill and Cayley, his treatment of elliptic functions was so different that indeed his development was entirely his own. Of course the person who could have resolved this issue was Hardy because Ramanujan was meeting him on a regular basis at Cambridge University. During discussions Hardy could have asked for the source of Ramanujan's ideas, but he did not. Hardy regrets not having asked such a question, but then says that at every meeting Ramanujan was presenting a dozen new identities. So Hardy did not ask Ramanujan how he got them because he spent all the time trying to understand these wonderful claims and prove them!

Infinite series: In the course on his investigations on the quintics, Abel was led to discuss the convergence of the binomial series. Abel realised the need to have a rigorous treatment of infinite series in general, and proceeded to provide a systematic treatment. After Abel, it was Cauchy who was the principal force in developing a theory to understand infinite series as we know today. Abel's name is associated with infinite series in many ways. There is the well known Abel summation formula. Also, there are results which are generally called *Abelian theorems*. An Abelian theorem is, roughly, one which asserts that, if a sequence or function behaves regularly, then some average of of the sequence or function behaves regularly. A method of summation is called regular, if it sums every convergent series to its ordinary sum. The interest in summability methods is that they provide a way to understand series which are not convergent. There are, for instance, important summability methods due to Abel, Euler, Cesaro and others. Generally, any theorem asserting the regularity of a summability method is called an Abelian theorem. The direct converse of an Abelian theorem is usually false, because if the regularity theorem for a summation method is reversible, then the method would be trivial since it applies only to convergent series. It is therefore important to obtain corrected forms of false converses to Abelian theorems, by attaching supplementary criteria. Such criteria are called Tauberian conditions, and the conditionally converse results, Tauberian theorems, named after the mathematician A. Tauber who first proved such a result.

Ramanujan's theory of infinite series: Ramanujan was a master in the manipulation and transformation of infinite processes, be they infinite series, continued fractions, integrals, or radicals. Sometimes Ramanujan did these operations formally, without rigorous proofs. Fortunately, he rarely made mistakes. He had his own *theory of constants*, whereby he would attach a value called *the constant of a series* to an infinite sum. Ramanujan's theory is related to the Euler-Maclaurin summation formula. One of his most startling claims is that the infinite sum 1+2+3+... of the positive integers equals -1/12. Ramanujan would show this result to his friends and teachers, who all said that this outrageous claim was definitely wrong, and that he should study the theory of infinite series before toying with them. We now realise that this claim of Ramanujan is very significant, because the series can be thought of formally as the representation of the Riemann zeta function at the point -1, but then the value of the Riemann zeta function at -1 is -1/12! This fact follows from a certain functional equation of the Riemann zeta function after the process of analytic continuation. It is amazing that Ramanujan obtained the correct value without a real understanding of complex variable theory and analytic continuation.

In connection with the rigorous treatment of infinite series Abel wrote, "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration." Fortunately, now, we understand infinite series very well, and are therefore able to appreciate the nature and importance of the work of Abel, Ramanujan, and others, from a proper perspective. Hardy's book on Divergent Series has a discussion both on Abelian theorems and Ramanujan's method.

The Paris Treatise: Although Abel published substantially in Crelle's Journal, he saved what he believed to be the best for the Paris Academy. Upon reaching Paris from Berlin, he worked on what would be called the Paris Treatise that he submitted to the Academy in October 1826. In this memoir, Abel obtained among other things, an important *addition theorem* for algebraic integrals. It is also in this treatise that we see the first occurence of the concept of the *genus* of an algebraic function. Cauchy and Legendre were appointed referees of this memoir. In Paris, Abel was disappointed to find little interest in his work, which he had saved for the Academy. He wrote to Holmboe "I showed the treatise to Mr. Cauchy, but he scarcely deigned to glance at it." Unfortunately, Abel's memoir was mislaid at thev Paris Academy and so Abel was convinced the work was lost forever. The disappointment of being ignored by both Gauss and Cauchy hit Abel hard. In this miserable state he encountered tuberculosis, and therefore decided to return to Norway. Crelle offered him an editorship of his journal, but Abel declined it. Those who provided the scholarship for Abel considered his trip a failure because he did not connect with either Gauss or Cauchy. So Abel's grant was not renewed and therefore he had to take a loan that he was never able to repay.

Around about that time, Jacobi came up with a paper on the transformation of elliptic integrals which was sent to Abel in 1828. Abel quickly demonstrated that Jacobi's results were consequences of his own. This motivated Abel to write several more papers, but his health was declining rapidly. He died on 6 April 1829.

Posthumous recognition: Ironically, two days after Abel's death, it was announced that Abel's Paris Treatise had been found, and it was praised as an outstanding work. Legendre saw the ideas of Abel and Jacobi and said "Through these works you two will be placed in the class of the foremost analysts of our times." The following year, the Paris Academy's Prize was posthumously awarded to Abel, with the money going to his mother in Gjerstad, Norway. Unaware of Abel's death, Crelle wrote to him saying that he finally succeeded in obtaining a permanent post for him in Berlin. Crelle said "As far as the future is concerned, you may now rest easy. You belong among us and will be secure. You will be coming to a good country, to a better climate, closer to science, and to sincere friends who appreciate you and are fond of you."

Recognition for Ramanujan: Although Ramanujan had his share of hardships in life, he was more fortunate than Abel in the sense that he was recognised for his contributions during his life time. When he wrote to Hardy, the British mathematician responded favourably. Ramanujan said in his second letter to Hardy "I find a friend in you who views my labours sympathetically." Concerned that Ramanujan may not live much longer, Hardy moved heaven and earth to get Ramanujan elected Fellow of the Royal Society (FRS) and Fellow of Trinity College in 1918.

The Abel Prize: In 1902, for the Abel Centenary, three tasks were specified. The

first was to arrange a cultural commemmoration. The second was to build a worthy monument in memory of the genius. The third was to establish an international prize. While the first two tasks were completed in 1902, the plans for the Abel Prize were set aside. The Nobel Prize awarded by Sweden, is generally acknowledged as the most prestigious prize in the sciences. Alfred Nobel announced his plans for the prize toward the end of the nineteenth century. However, there is no Nobel Prize in mathematics, and this actually prompted Sophus Lie, a world reknowned Norwegian mathematician, to lobby for a prize in the name of Abel for mathematics. Unfortunately, Lie died in 1899, and so the effort to create the Abel Prize did not materialise.

The most prestigious prize for mathematics has been the Fields Medal awarded at the International Congress of Mathematicians which is held once every four years. Two to four Fields medals are awarded at each of these congresses to mathematicians under the age of forty. The Fields Medal carries only a cash prize of 1,500 Canadian dollars and so does not match the Nobel Prize in prize money. On August 23, 2001, a year prior to the Abel Bi-Centenary, it was announced that the Norwegian Government would establish an Abel Fund worth NOK 200 million. Part of the fund's money would be used by the Norwegian Academy of Science and Letters to award an annual *Abel Prize* for mathematics worth about $750,000, which is of the order of magnitude of the Nobel Prize, and a long awaited one. Unlike the Fields Medal, the Abel Prize is for lifelong contributions to mathematics. Jacob Palais, President of the International Mathematical Union said in 2002, that the Abel Prize will alter the global mathematical landscape and raise the visibility of mathematics in society. The first Abel Prize was awarded in March 2002 to Jean-Pierre Serre of France. Serre had won the Fields Medal in 1954 as a young mathematician. The second Abel Prize awarded in March 2004 was shared by Michael Atiyah of Oxford University and I. M. Singer of MIT. Atiyah had won the Fields Medal in 1966.

The Ramanujan Prize: Just this year, the Abdus Salam International Centre for Theoretical Physics (ICTP) announced the creation of a *Ramanujan Prize* of $10,000 to be awarded annually to a mathematician under 45 of the Third World. When Abdus Salam created the ICTP in the early sixties, it was with the intention of strengthening science in the Third World. Thus, it is not surprising that the Ramanujan Prize created by the ICTP is for mathematicians from Third World countries. Interestingly, the Ramanujan Prize of the ICTP is also supported by the Abel Foundation.

A game of youth: Hardy has stressed that the greatest mathematicians made their most significant discoveries when they were very young. He pointed out that Galois who died at 18, Abel at 29, and Riemann at 40, had actually made their mark in history. So Hardy said that the real tragedy of Ramanujan was not his early death at the age of 32, but that in his most formative years, he did not receive proper training, and so a significant part of his work was rediscovery. It is with this same view that it has been argued that with the age restriction for Fields Medals, one is not missing out much in recognising outstanding mathematicians. But the Abel Prize will recognise mathematicians of all ages.

Leopold von Crelle said that Abel is one of those rare beings that nature produces barely once a century. Such high praise applies equally well to Ramanujan. There is much for us to learn from the lives of Abel and Ramanujan. In particular, we should aim to follow their example and not let anything stand in the way of lofty intellectual pursuit.