Elliptic Elegance

If we prefer the hypotenuse and side as coordinates to the Cartesian, we are led to the incredible but logical conclusion that the ellipse is a Lorentz Transform of the circle with suitable representation of variables. For a hundred years there has been an 'addiction' to the hyperbola for the obvious reason we deal with a difference of squares in Special Relativity.

In an ellipse with eccentricity e and major and minor axes 2a and 2b respectively, if we take a point on the ellipse we notice that the distance from the focus is the diagonal, the projection on the x axis, the side and the y coordinate the invariant as we move from one ellipse to another keeping the minor axis fixed and changing the eccentricity.

A circle is a degenerate form of an ellipse with distance between foci =0, ie. e =0. Since there are 2 foci for an ellipse, there are two distance from the point and two projections. The sum of these distances is equal to the major axis 2a and the sum of the projections is the distance between the foci 2ae. Therefore the point on the ellipse represents 2 particles with total energy 2a and total momentum 2ae.

Likewise in the circle a point on the circumference represents 2 particles with equal energies (radius) and equal and opposite momenta the projections on the x axis. The total momentum is zero since the foci coincide. If the y coordinate of the point on the circle is the same as that of the point on the ellipse we note that (r_1,x_1) and (r_2,x_2) are the Lorentz Transforms of (b,x) and (b,-x) as shown in the diagrams which are self-explanatory

We can write

$$x = \frac{t_0 V}{\sqrt{1-V^2}}, \quad t = \frac{t_0}{\sqrt{1-V^2}} \quad \text{if } \frac{x}{t} = V$$

Similarly

$$x' = \frac{t_0 V'}{\sqrt{1-V'^2}}, \quad t' = \frac{t_0}{\sqrt{1-V'^2}} \quad \frac{x'}{t'} = V'$$

$$\frac{V^1}{\sqrt{1-V^{1^2}}} = \frac{V}{\sqrt{1-V^2}\sqrt{1-v^2}} - \frac{V}{\sqrt{1-V^2}\sqrt{1-v^2}}$$

simplying

$$V^1 = \frac{V-v}{1-Vv}$$

This is just what Einstein has done for Energy and momentum which behave like time and space (c = 1). He writes them in terms of rest mass m_0 which corresponds to t_0.

The Lorentz Transformation is derived using the [illegible] approach to special relativity of the author. The Trigonometric relation is then the Lorentz Transformation.

<u>Note</u> For space-like intervals $x^2 - t^2 = x_0^2 > D$

We write $x = x_0 \sec\theta, \; t = x_0 \tan\theta$

$x^1 = x_0 \sec\theta^1, \; t^1 = x_0 \tan\theta^1$

$$\frac{x}{t} = \frac{1}{\sin\theta} = \frac{1}{V} > 1, \quad \frac{x^1}{t^1} = \frac{1}{\sin\theta^1} = \frac{1}{V^1} > 1$$

But $v = \sin\theta_1 < 1$.

This feature has been discussed in detail by the author in the study of velocity transformation. When the ratio is >1 x represents the distance between two particles. The ratio has the transformation properties of a velocity but is not a velocity. This renders the concept of a Tachyon, faster than light particle meaningless.

Science section – The Hindu, 24 October 2002.

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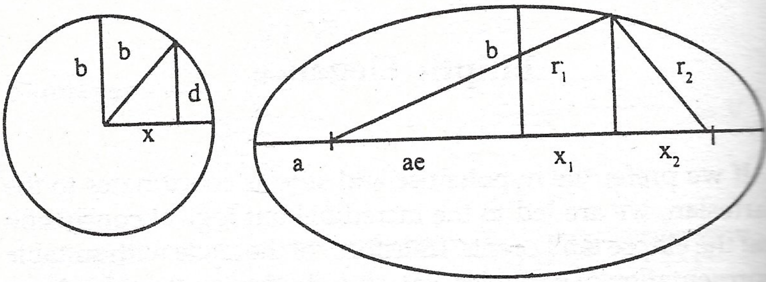


Major axis = 2a, Minor axis = 2b. Focal distance 2ae

$a = \frac{b}{\sqrt{1-e^2}}$ e = eccentricity = Velocity (with velocity of light = 1)

$b^2 - x^2 = d^2 = r_1^2 - x_1^2 = r_2^2 - x_2^2$

$r_1 + r_2 = 2a, \; x_1 + x_2 = 2ae$

(Note if $x_1 > 2ae$, x_2 is negative.

Lorentz Transformatrix matrix

$$L(e) = \begin{bmatrix} \frac{1}{\sqrt{1-e^2}} & \frac{e}{\sqrt{1-e^2}} \\ \frac{e}{\sqrt{1-e^2}} & \frac{1}{\sqrt{1-e^2}} \end{bmatrix}$$

transforms (b,x), $(b,-x)$ to (r_1,x_1) and (r_2,x_2).

If $d=b$, $r_1=r_2=a$ $\quad x_1=x_2=ae$

and if we define

$e = \sin\theta$, $\quad \frac{x_1}{r_1} = \sin\theta_1$, $\quad \frac{x_2}{r_2} = \sin\theta_2$, $\quad \frac{x}{b} = \sin$

then we obtain the sec θ, tan θ representation of section

Corollary I

Since the ellipse is a Lorentz transform of a circle with radius equal to the minor axis, ellipses with the same minor axis are Lorentz Transforms of one another. With 'relative eccentricities' obeying the velocity Transformation law of special relativity...

Corollary II

If we replace e by $-e$ in $\angle(e)$ we find $\angle(-e)$ transforms (b,x), $(b,-x)$ to $(r_2,-x_2)$ and $(r_1,-x_1)$ ie. interchanging r_1 and r_2 and changing the sign of their projections. Consequently the distance between the foci is $-(x_1+x_2) = -2ae$. We get the same ellipse with focal distance negative ie. the foci get 'switched'. This is perfectly consistent with the interpretation of 'relative velocity' between $-e$ and $+e$ as $e^1 = \frac{2e}{1+e^2}$. $\angle(e^1)$ transforms $(r_1,-x_1)$ to (r_2,x_2) and $(r_2,-x_2)$ to (r_1,x_1). The inverse of $\angle(e1)$ is $\angle(-e^1)$ which performs the inverse transformations.

We note that if we define $\frac{x_1}{r_1} = e_1$ and $\frac{x_2}{r_2} = e_2$

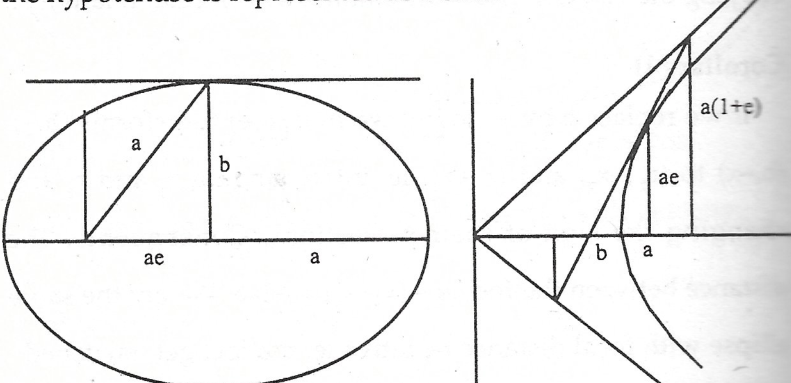
$$e^1 = \frac{2e}{1+e^2} = \frac{e_1+e_2}{1+e_1e_2}$$

$$\angle(e^1) = \angle(e)\angle(e) = \angle(e_1) = \angle(e_2) = \angle(e_2)\angle(e_1)$$

Operation (e_1) on (r_1-x_1) transforms it to (d,o) and (e_2) on (d,o) transforms it to (r_2,x_2)

Hyperbolic elegance

We now point out a one-to-one correspondence between the elliptic and hyperbolic representations noting that in the latter the hypotenuse is represented as a side.



Ellipse with minor axis b and eccentricity e becomes a tangent to the hyperbola $x^2 - y^2 = b^2$ at (x, y) such that $\frac{y}{x} = e$ and slope $\frac{dy}{dx} = \frac{x}{y} = \frac{1}{e}$, x corresponds to the hypotenuse a, y to ae with b as the intercept of the hyperbola on the x axis. Circle of radius b is the tangent at (b,o) to the hyperbola and is parallel to y axis.

The points of intersection of the tangent with the asymptotic lines have y coordinates a(1+e) and –a(1-e) ie the distance of a focus from the extremities.

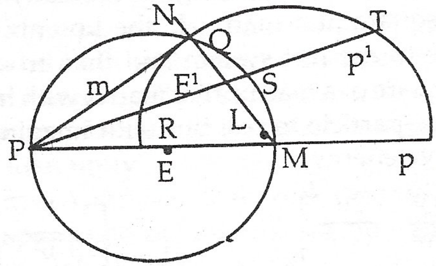
The two points of intersection of the tangent with the hyperbola $x^2 = y^2 = d^2$, $d<b$ have as their projections on the x axis, r_1 and r_2 in the elliptical representation since they represent the intersection of the line parallel to the x-axis with the ellipse. The y coordinates in the hyperbolic representation are the projections x_1 and x_2 of r_1 and r_2 in the elliptic representation. Their sum is the focal distance 2ae while $r_1+r_2 = 2a$.

The equation for the tangent ey = x-l is the polar equation of the ellipse with x=r, y=r cos θ.

Is this not mathematical ecstasy?

Circular elegance

We now demonstrate that circles can be a striking representtion of Lorent Transformation. To facilitate identification we use the relativistic variables E (energy) and p (m omentum), m (restmass) and set the velocity of light to unity.



Draw a circle with diameter PM = E, and a circle with centre M and radius p intersecting the first at Q. Draw a line from P intersecting the first circle at S and the second at R and T.

$PQ^2 = m^2 = PM^2 - MQ^2 = E^2 - p^2 = PR.\ P,T$
$= PS^2 - ST^2 = E^{1^2} - p^2 = (E^1 + p^1)\ (E^1 - p^1)$
S is the mid point of RT, $RS = ST = p^1$, $PS = E^1$

Thus, E^1 and p^1 are Lorentz Transforms of E and p, In turn if we draw cirlce with S as centre and p1 as radius it will intersect a third circle with Pas centre and m as radius at N such that NP is a tangent to the circle of radius p^1 at N, SN and QM are tangents to the circle with radius m the circle with PS as diameter passes through N.

To establish connection with Trigonometric elegance of Apendix II, we take a point L on QM such $QL = p^1$. Then $PL = PS = E^1$ and we compare triangles PQM and PQL as in Appendix II

The Fourfold elegance leads to the refrain:

God created the line, the ellipse, the hyperbola and the circle.
Man understands the connection through the Lorentz Transformation.

Lorentz Transform in the Twenty-First Century

If $\mathcal{L}$ is a Lorentz Matrix so is $-\mathcal{L}$. If $\mathcal{L}$ reverses momentum keeping energy constant, $-\mathcal{L}$ reverses energy keeping momentum constant, both reversing the velocity v. If $\mathcal{L}$ is expressed as the square of a matrix **L**, the Lorentz matrix **L** brings the particle to the rest system and then to -v. If -L is expressed as the square of a matrix, that matrix with imaginary elements brings the particle to rest but with imaginary mass and then to negative energy.

$$\mathcal{L}(\upsilon) = L^2 = \begin{bmatrix} \dfrac{1+\upsilon^2}{1-\upsilon^2} & \dfrac{-2\upsilon}{1+\upsilon^2} \\[2ex] \dfrac{-2\upsilon}{1+\upsilon^2} & \dfrac{1+\upsilon^2}{1-\upsilon^2} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} \dfrac{1}{\sqrt{1-\upsilon^2}} & \dfrac{-\upsilon}{\sqrt{1-\upsilon^2}} \\[2ex] \dfrac{-\upsilon}{\sqrt{1-\upsilon^2}} & \dfrac{1}{\sqrt{1-\upsilon^2}} \end{bmatrix}$$

$$-\mathcal{L} = (i\,L)^2, \quad \mathcal{L}(\upsilon) = L^2(v) = L(V), \; V = \frac{2\upsilon}{1+\upsilon^2}, \quad \upsilon < 1, \; V < 1$$

$$\text{Note: } \mathcal{L}\left(\tfrac{1}{\upsilon}\right) = -\mathcal{L}(\upsilon), \qquad L\left(\tfrac{1}{\upsilon}\right) = -\,iL^*, \; \upsilon < 1, \; \tfrac{1}{\upsilon} > 1$$

$$L^* = L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad L^2 = (L^*)^2$$

L^* is a semi Lorentz matrix reversing the difference of squares.

What is the physical meaning L^* and iL^* ?

The properties of iL and L^* are connected to the striking features of the velocity transformation formula in which either all three velocities are less than unity or two of them greater than and one less than unity. This has been clarified by the author's New Rod Approach to Special Relativity which explains the distinction between space-like and time-like